EDS 212: Day 2, Lecture 1

Limits and derivatives

August 4th, 2025

Overview of Day 2 topics

- Limits
- Derivatives
- Integrals
- Differential equations (just a bit)

Limits

Limits

Definition

We say a number L is the limit of a function $f(x)$ as its variable x approaches a number c, if the function's output values $f(x)$ approach L when x get closer to c.

We write this symbolically as:

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} f(x) = L.$$

We read this as: "The limit of $f(x)$ as x approaches c is L."

* Make this into a blank slide that I completely fill in.

Limits example: numerical

What is the limit of $f(x) = 2x^2 - 4$ as x approaches 2?

We need to examine the values of $f(x)$ as x gets closer to 2 from both sides.

x approaches 2 from negative direction

x approaches 2 from positive direction

x	1.5	1.75	1.95	1.99	**2.0**	2.01	2.05	2.25	2.5
f(x)	0.5 $= f(1.5)$	2.12 $= f(1.75)$	3.60	3.92	**4.0**	4.08	4.41	6.13	8.5

f(x) approaches 4 from negative direction

f(x) approaches 4 from positive direction

From both directions it looks like $f(x)$ converges to 4.

So, we say that "The limit of $2x^2 - 4$ as x approaches 2 is 4". We can write it like this:

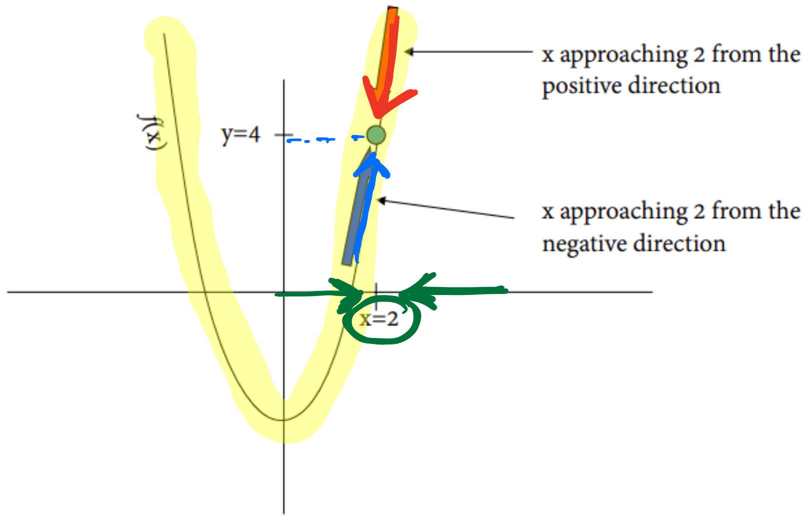
$$\lim_{x \to 2}(2x^2 - 4) = 4$$

$$\lim_{x \to 2} 2x^2 - 4 = 4.$$

* Break this up into separate slides.

Limits example: graph

What is the limit of $f(x) = 2x^2 - 4$ as x approaches 2?



From both directions it looks like $f(x)$ converges to 4.

So, $\lim_{x \to 2}(2x^2 - 4) = 4$.

*This one needs to have $f(x) = \frac{|x-2|}{x-2}$ defined. Maybe break down exercise

Do limits always exist?

The function $|x|$ is the **absolute value function**. It is such that

$|2| = 2$

$|-e| = e$

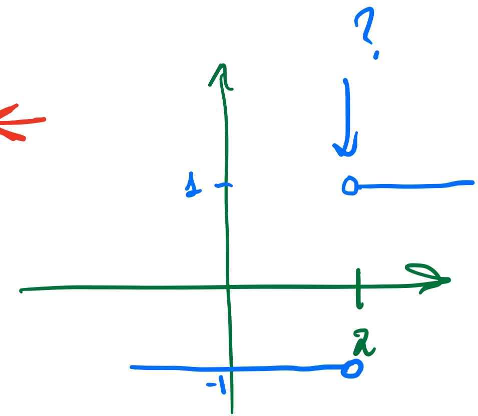
$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

✏️ What happens to $f(x)$ at $x = 2$?

$f(x) = \frac{|x-2|}{x-2}$

✏️ Try calculating:

$$\lim_{x \to 2} \frac{|x-2|}{x-2}.$$



Hint: Try to do numerical approximations or draw the graph of this function.

As $x \to 2$ with $x < 2$, we always get $f(x) = -1$.

As $x \to 2$ with $x > 2$, we always get $f(x) = 1$.

$f(2) = |2-2| = 0$ = undefined.

$2-2$ 0

Do limits always exist?

✏️ What happens to $f(x)$ at $x = 2$?

This function is not defined at $x = 2$ since it would require us to divide by zero, which is undefined.

Try calculating $\lim_{x \to 2} \frac{|x-2|}{x-2}$.

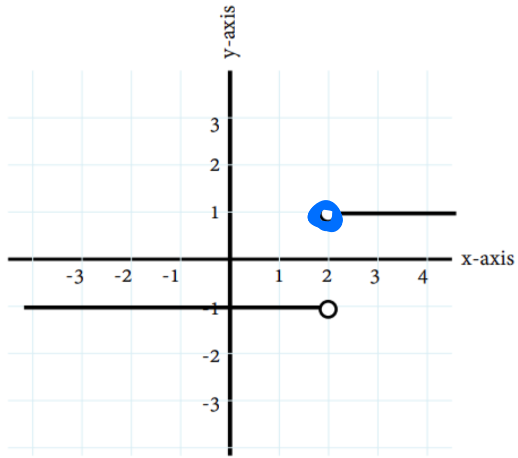
Let's investigate:

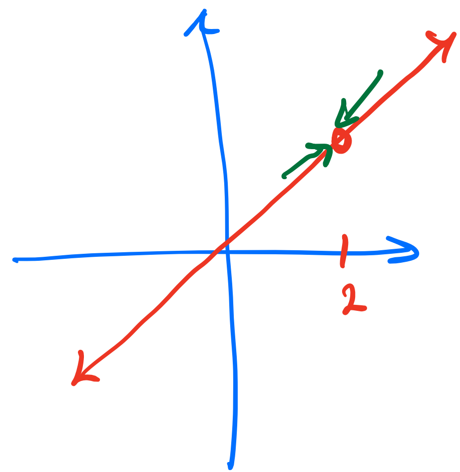
x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	-1	-1	-1	?	1	1	1

Do limits always exist?

The function $\frac{|x-2|}{x-2}$ approaches different values from either side at $x = 2$.

$f(x) = x,\ x \neq 2$



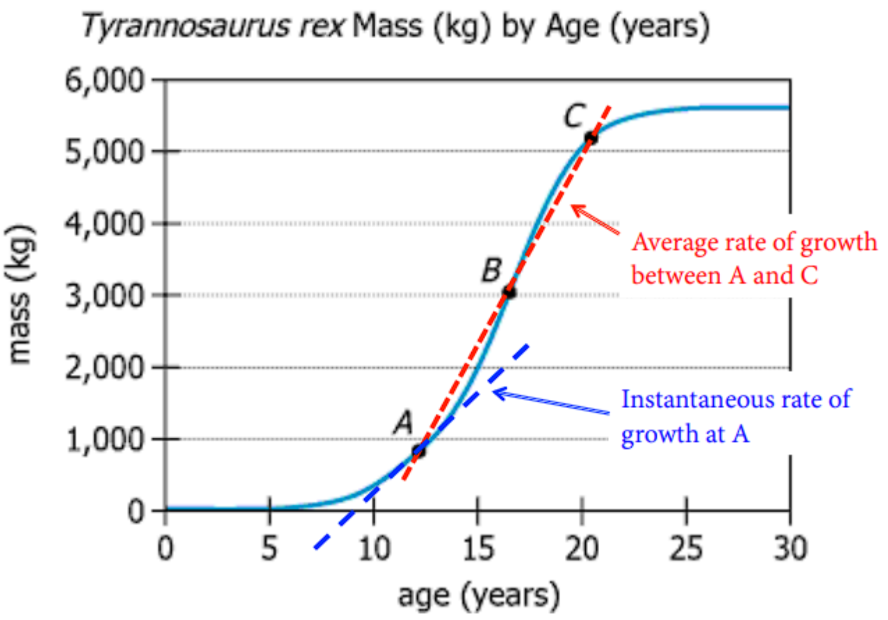


Therefore… the limit does not exist!

* Maybe we don't go over this?

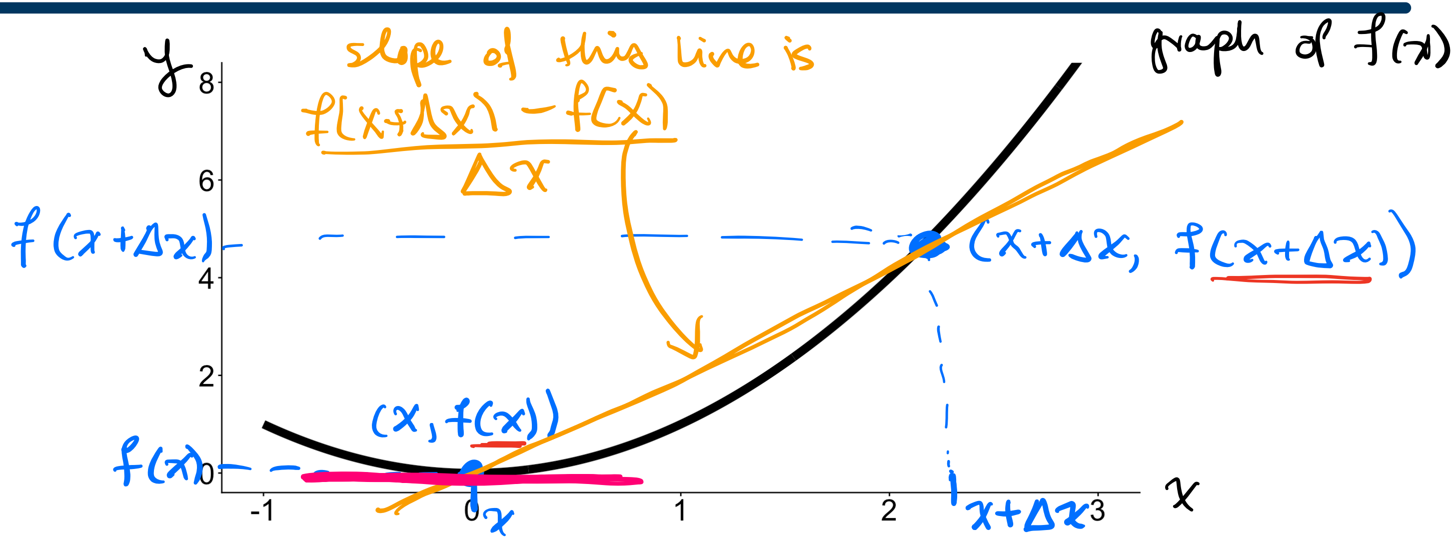
Introduction to Derivatives

Recall average and instantaneous rate of change



Big idea: Taking the average rate of change to a set limit will eventually converge to the instantaneous.

Average slope



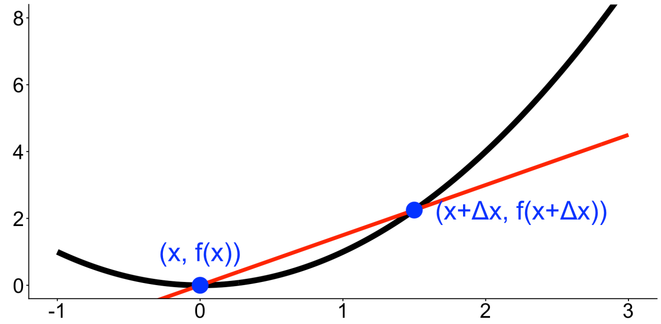
Start with: a point on the graph $(x, f(x))$

Δx = represents a short distance on the x axis

Nearby point: is $(x+\Delta x, f(x+\Delta x))$

Average slope: $= \dfrac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Average slope



Start with a point $(x, f(x))$ on the graph.

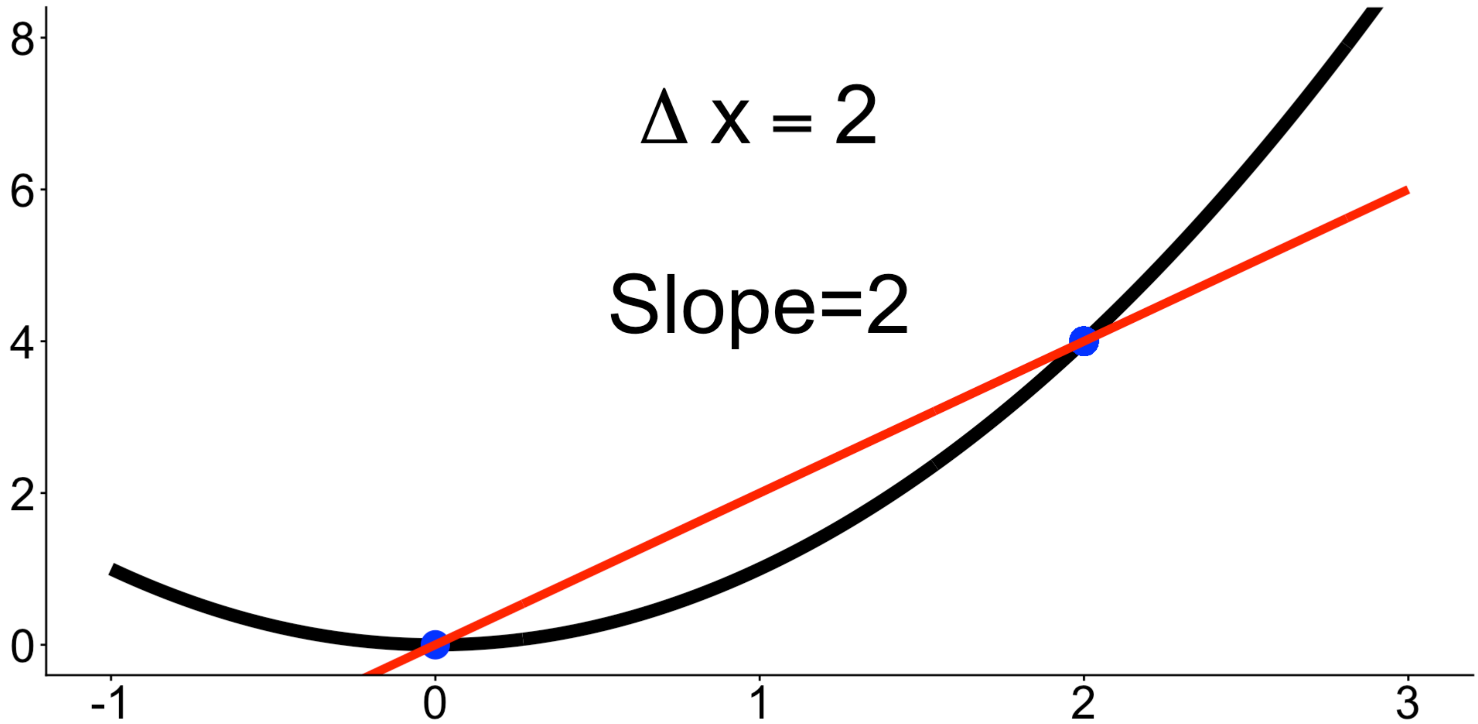
A nearby point can be $(x + \Delta x, f(x + \Delta x))$, where Δx is a short distance from x.

The average slope between $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ is given by

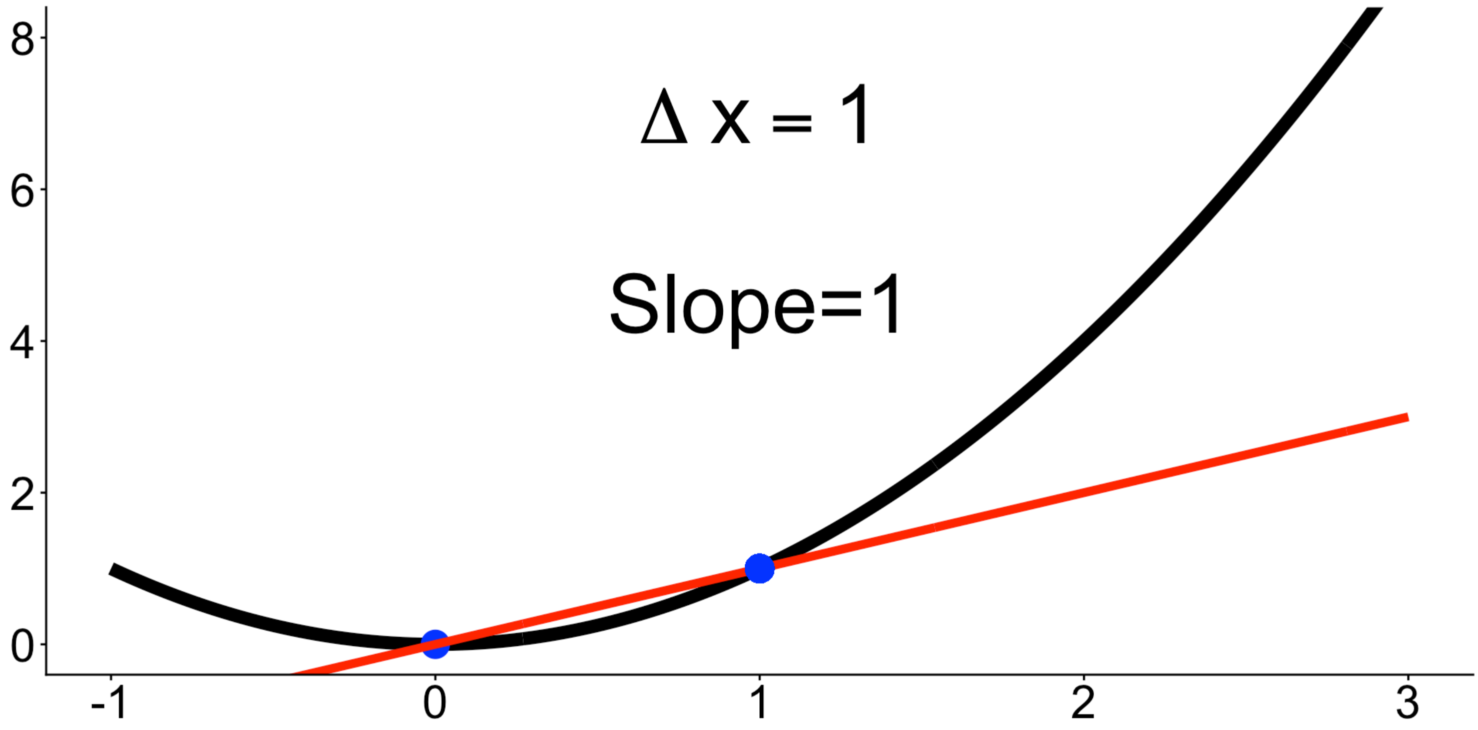
$$\text{average slope} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

What happens when Δx becomes smaller and smaller?

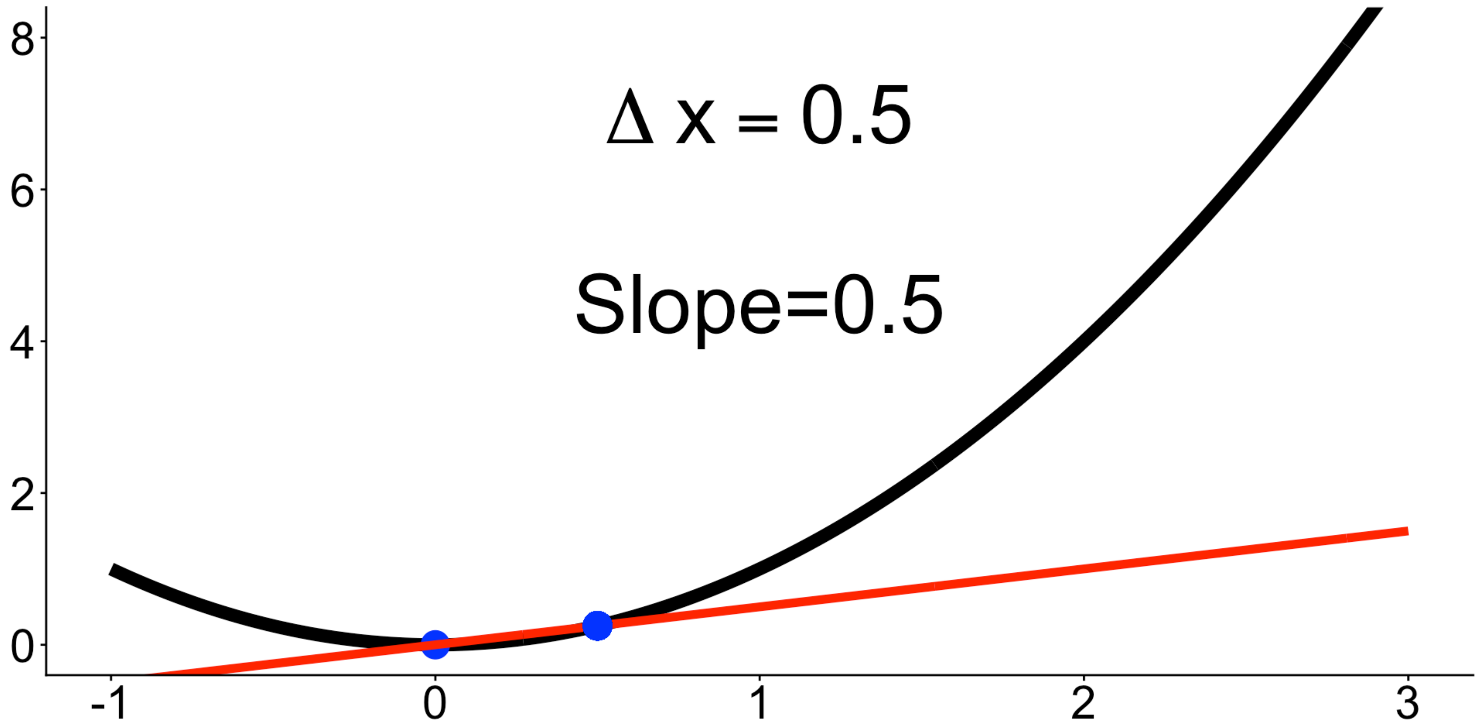
What happens when $\Delta x \to 0$?



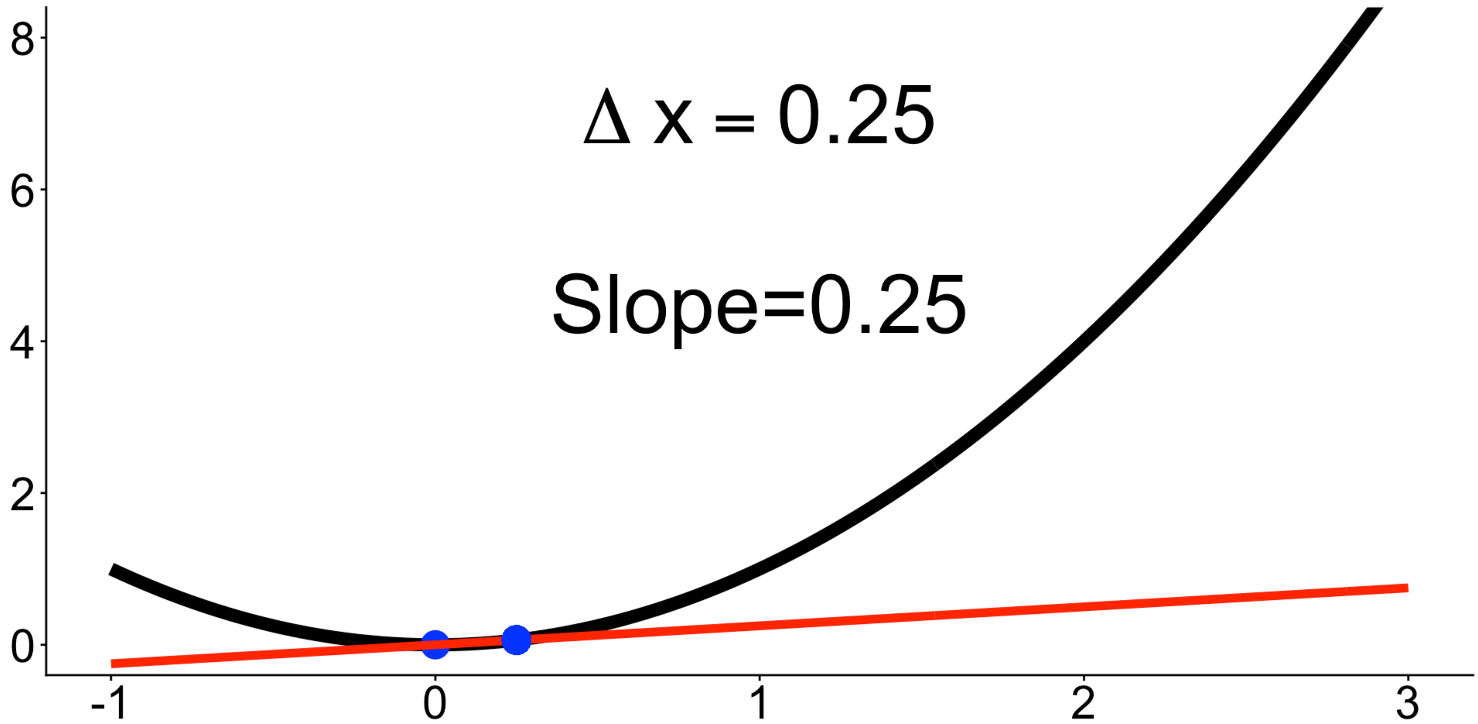
What happens when $\Delta x \to 0$?



What happens when $\Delta x \to 0$?



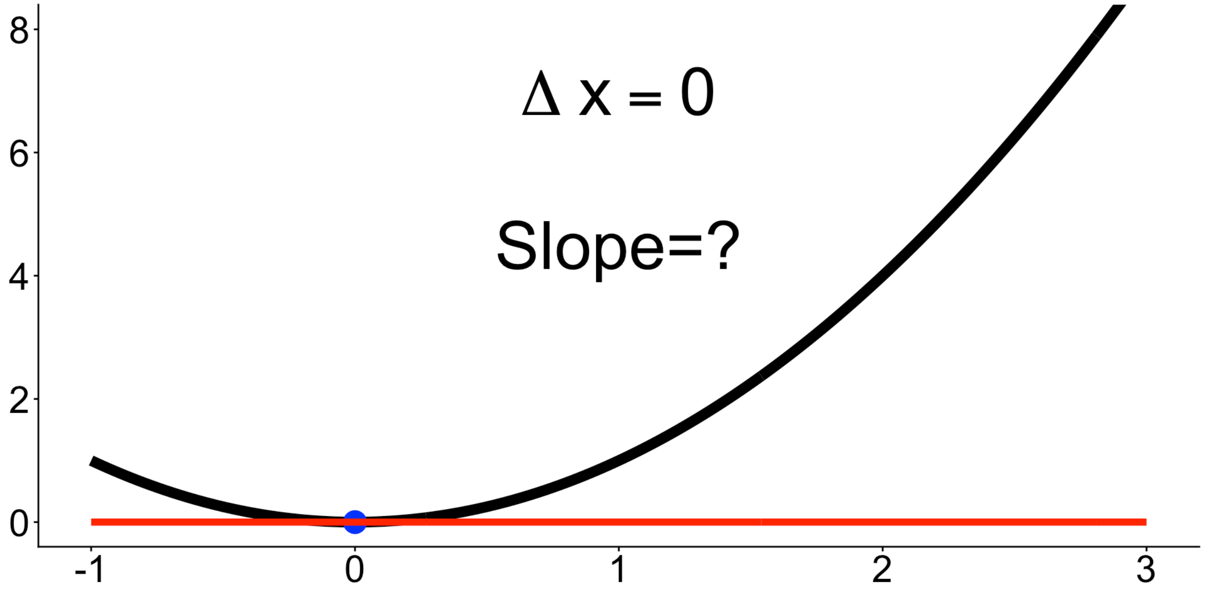
What happens when $\Delta x \to 0$?



Tangent lines

What if we let $\Delta x = 0$?

Then we would have a slope line that only touches the graph at exactly x.



These very special types of lines are called **tangent lines**.

Derivative definition

Remember the slope of the line that passes between two points (x_1, y_1) and (x_2, y_2) is:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If we have a function $f(x)$, then $(x, f(x))$ represents a point on its graph.

Choose a different point on the graph that is Δx away: $(x + \Delta x, f(x + \Delta x))$.

Plug these pointson the graph of $f(x)$ into the slope equation:

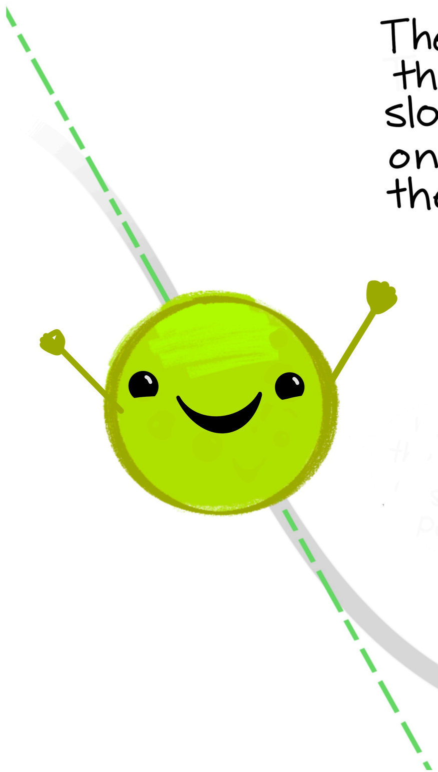
$$\text{slope} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

When we let $\Delta x \to 0$ we get **the slope of the tangent line at x**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This is the derivative of f at x. Also denoted $\frac{df}{dx}$.

$\frac{df}{dx} =$



The expression for the instantaneous slope at any point on a function, aka the **derivative**

IS FOUND BY:

① Finding an expression for the **slope** between 2 points separated by Δx...

$$\frac{df}{dx} = \underbrace{\lim_{\Delta x \to 0}}_{②} \overbrace{\frac{f(x+\Delta x)-f(x)}{\Delta x}}^{①}$$

② & evaluating that slope as the points get infinitely close together.

* Do we want to keep this as an exercise or maybe have them calculate with the definition?

Not all functions are differentiable

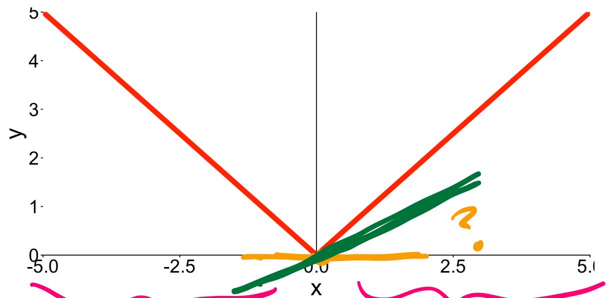
Or maybe we don't cover it.

- All differentiable functions are continuous, but not all continuous functions are differentiable.

- The absolute value function $|x|$ is one example, this is defined by

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases} .$$

It's graph is:



← graph of $|x|$

equal to $-x$

on the left side equal to x

Let's take a 5 minute break

Meet back at 11 am.

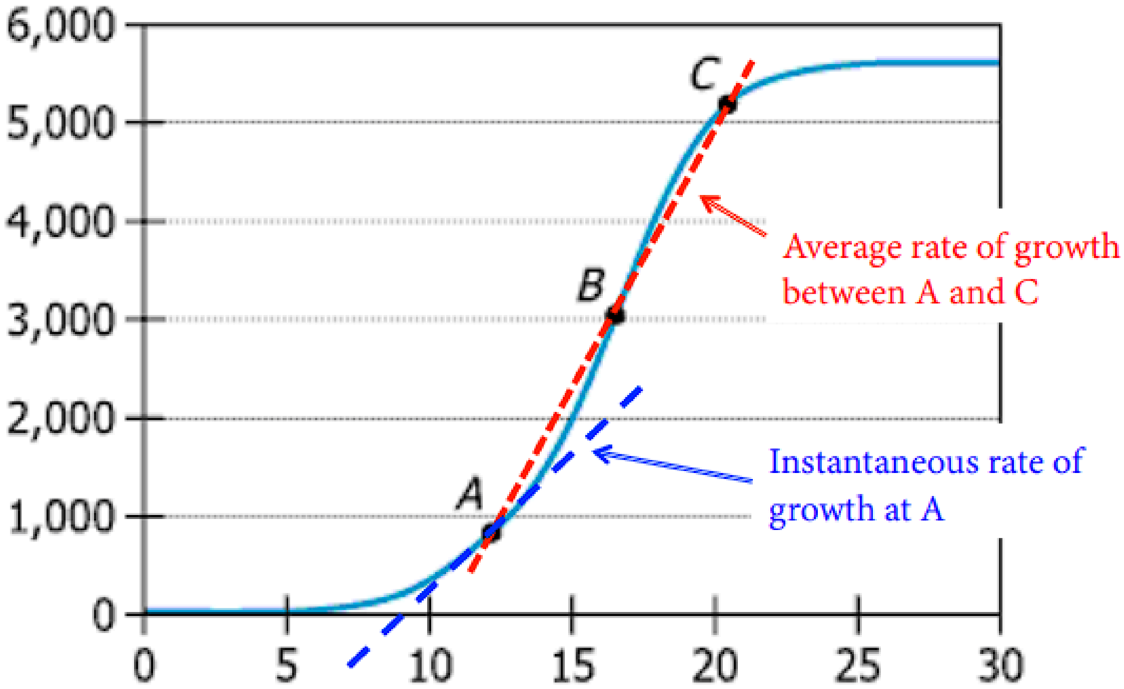
image: Flaticon.com

* Move after rate of change in EDS.

Why derivatives in an EDS program?

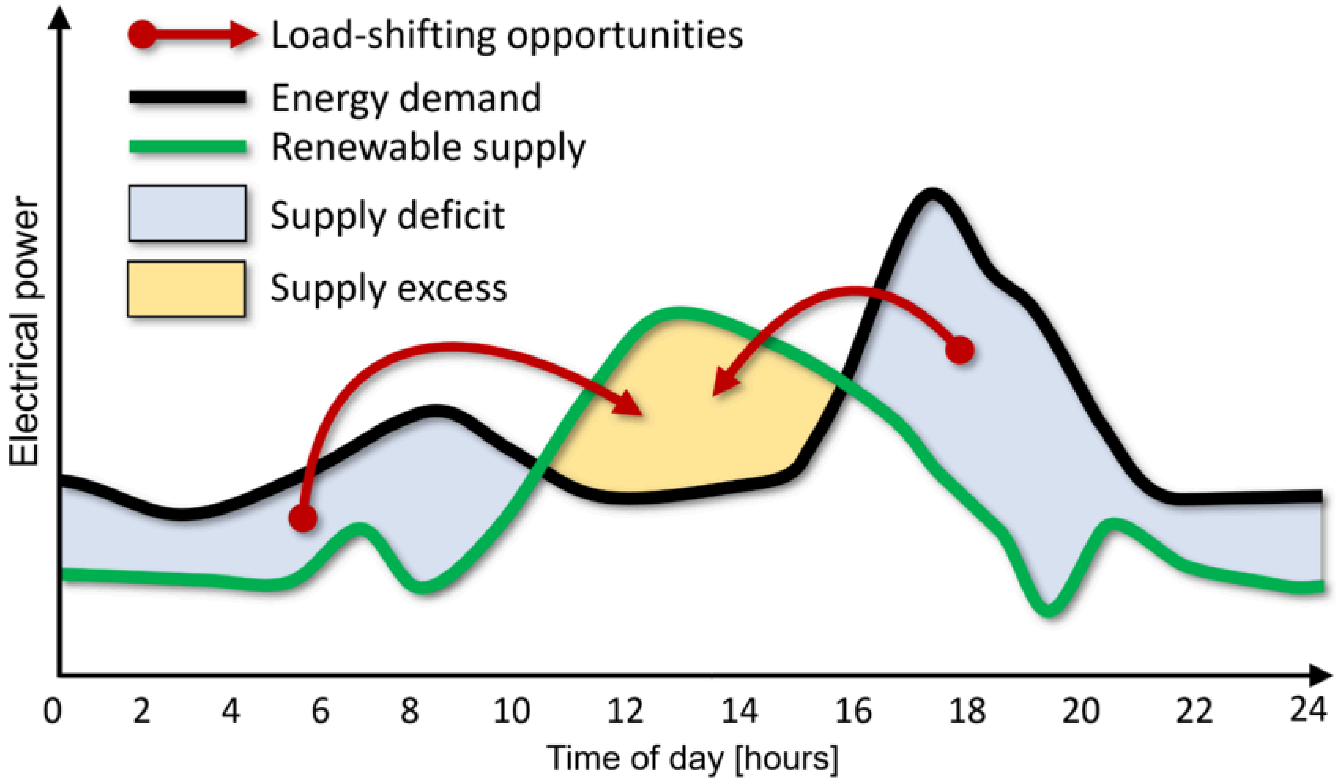
Describing rates of change is common in environmental science (rate of pollutant concentration change, rate of population growth, rate of energy consumption)

Calculus and derivatives are used to study change



Environmental Science studies change

Environmental Science studies change



Environmental Science studies change

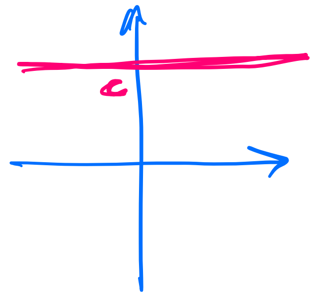
🤔 Talk with a partner, what are 2 fields of environmental science where studying the rate of change and derivatives would be important?

* Move break after here.

Rules for differentiation (1)

Constant Rule ✏️

Let c be a constant (a number, not a variable).



If $f(x) = c$, then $f'(x) = 0$.

$$\frac{d}{dx} c = 0.$$

Power Rule ✏️

Let n be a number, if

$f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Same: $$\frac{d}{dx} x^n = nx^{n-1}.$$

Rules for differentiation (1)

Constant Rule

Let c be a constant (so, a number). If $f(x) = c$, then $f'(x) = 0$. Equivalently

$$\frac{d}{dx}c = 0$$

Power Rule

Let n be a (real) number, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Equivalently

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Examples ✏️

$$y = 100 \qquad f(x) = x^5 \qquad f(x) = \frac{1}{x^2}$$

Rules for differentiation (2)

Sum and Difference Rules

✏️ Let $f(x)$ and $g(x)$ be differentiable functions.

Then

$$(f+g)' = f' + g' \quad \leftarrow \text{concise notation}$$

and

$$(f(x) - g(x))' = f'(x) - g'(x).$$

Example

$$f(x) = x^3 - x^2 - 15 \quad \text{constant} \Rightarrow \text{derivative is } 0$$

$$f'(x) = 3x^{3-1} - 2x^{2-1} + 0 = 3x^2 - 2x.$$

Rules for differentiation (2)

Sum and Difference Rules

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

😵 *All this says is: if the function has pieces that are added or subtracted you can take the derivative of each individual piece and add those derivatives.*

Example ✏️

$$f(x) = x^3 - x^2 - 15$$

* Can change this into $f' + g'$ notation.

Rules for differentiation (3)

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Constant Multiplier Rule

✏️ The derivative of a constant c multiplied by a function f(x) is the same as the constant multiplying the derivative:

$$(c f(x))' = c f'(x).$$

$$\frac{d}{dx} c f(x) = c \frac{d}{dx} f(x)$$

$$\frac{d}{dx} -\frac{8}{x} = -\frac{d}{dx}\frac{8}{x} = -\frac{d}{dx} 8x^{-1} = -8\frac{d}{dx}x^{-1} = (-8)(-1)x^{-2} = 8x^{-2} = \frac{8}{x^2}.$$

Example:

$$y = x^2 + \sqrt{2}x^1 - \frac{8}{x} + 4 = x^2 + \sqrt{2}x - 8x^{-1} + 4$$

$$y' = 2x + \sqrt{2}x^{1-1} - (-1)8x^{-1-1} + 0 = 2x + \sqrt{2} + 8x^{-2}$$

Rules for differentiation (3)

Constant Multiplier Rule

The derivative of a constant c multiplied by a function $f(x)$ is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}f(x)$$

Example ✏️

$$y = x^2 + \sqrt{2}x - \frac{8}{x} + 4$$

Derivative of logs & exponents

Two special derivatives:

- $\frac{d}{dx}(e^x) = e^x$

- $\frac{d}{dx}ln(x) = \frac{1}{x}$

Summary of differentiation rules

Differentiation Rule	$f'(x)$ notation	$\frac{d}{dx}$ notation
Constant Rule	$f(x) = c \Rightarrow f'(x) = 0$	$\frac{d}{dx}[c] = 0$
Power Rule	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$	$\frac{d}{dx}[x^n] = nx^{n-1}$
Sum Rule	$(g(x) + h(x))' = g'(x) + h'(x)$	$\frac{d}{dx}[g(x) + h(x)] = \frac{d}{dx}g(x) + \frac{d}{dx}h(x)$
Difference Rule	$(g(x) - h(x))' = g'(x) - h'(x)$	$\frac{d}{dx}[g(x) - h(x)] = \frac{d}{dx}g(x) - \frac{d}{dx}h(x)$
Constant Multiplier Rule	$(c \cdot g(x))' = c\, g'(x)$	$\frac{d}{dx}[c\, g(x)] = c\, \frac{d}{dx}g(x)$

And two special derivatives: $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}ln(x) = \frac{1}{x}$.

There's more differentiation rules (product, quotient, chain rule).

They're fun, but we won't be reviewing them.

Higher order derivatives

- We can continue to take derivatives of derivatives.
- The second derivative becomes the rate of change of the rate of change.
- If we interpret the first derivative as velocity, then the second derivative is acceleration.

- Notation:
  - Function: $f(x) = y$
  - 1st derivative: $f'(x)$ or $\frac{dy}{dx}$
  - 2nd derivative: $f''(x)$ or $\frac{d^2y}{dx^2}$
  - nth derivative: $f^{(n)}(x)$ or $\frac{d^ny}{dx^n}$

Derivatives practice

Solve individually and then check with a partner.

1. Which rules should you use to take these derivatives?

A) $f(x) = 3x^4$ B) $y = 4x^2 + 3x - 16$

2. Find the derivatives of these functions

A) $y = 3x^2$ B) $h(x) = 2x(x^2 + 3x)$ C) $g(y) = 2\sqrt{y} - e^y + 20\ln(y)$

3. Find the 3rd derivative of:

$$G(z) = 3z^4 - 8z^3 + 2z - 19$$

A) $y' = 3 \cdot 2x^{2-1}$

$= \underline{\underline{6x}}$.

Derivatives practice

✏️ Let's see a solution!

2.B) Find the derivative

$$h(x) = 2x(x^2 + 3x)$$

$$h(x) = 2x^3 + 6x^2$$

$$h'(x) = 6x^2 + 12x$$

Derivatives practice

✏️ Let's see a solution!

2.C) Find the derivative

$$g(y) = 2\sqrt{y} - e^y + 20\ln(y)$$

$$g(y) = 2y^{1/2} - e^y + 20\ln(y)$$

$$g'(y) = 2 \cdot \frac{1}{2} y^{1/2 - 1} - e^y + 20 \cdot \frac{1}{y}$$

$$g'(y) = y^{-1/2} - e^y + \frac{20}{y}$$

$$g'(y) = \frac{1}{\sqrt{y}} - e^y + \frac{20}{y}$$

Derivatives practice

✏️ Let's see a solution!

3. Find the 3rd derivative of $G(z) = 3z^4 - 8z^3 + 2z - 19$.

$$G'(z) = 12z^3 - 24z^2 + 2 + 0$$

$$G''(z) = 36z^2 - 48z + 0$$

$$G'''(z) = 72z - 48$$

* Add solutions in a separate slide.

Partial derivatives

Partial derivatives

When we find a partial derivative, we find an expression for the slope with respect to *one variable* in a multivariate function.

Notation: the partials of $f(x, y, z)$ are $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$

We read $\frac{\partial f}{\partial x}$ as "the partial derivative of f with respect to x"

How to take partial derivatives? Find the derivative with respect to a single variable, *treating all others as constants.*

Partial derivatives example

✏️ Find all partial derivatives of

Key: treat all other variables as constants.

$$B(x,y,z) = 0.4x^3y - 3.6y^2 + 4zx + 8$$

$$\frac{\partial B}{\partial x} = 0.4y(3x^2) + 0 + 4z + 0$$

$$= 1.2x^2y + 4z$$

$$\frac{\partial B}{\partial y} = 0.4x^3 - 7.2y + 0 = 0.4x^3 - 7.2y$$

$$\frac{\partial B}{\partial z} = 0 + 4x + 0 = 4x.$$

Partial derivatives example

Find all partial derivatives of

$$B(x, y, z) = 0.4x^3 y - 3.6y^2 + 4zx + 8$$

$$\frac{\partial B}{\partial x} = 1.2x^2 y + 4z$$

$$\frac{\partial B}{\partial y} = 0.4x^3 - 7.2y$$

$$\frac{\partial B}{\partial z} = 4x$$

* merge this with next slide.

Partial derivatives example

Find all partial derivatives of

$$B(x, y, z) = 0.4x^3y - 3.6y^2 + 4zx + 8$$

- $$\frac{\partial B}{\partial x} = 1.2x^2y + 4z$$
- $$\frac{\partial B}{\partial y} = 0.4x^3 - 7.2y$$
- $$\frac{\partial B}{\partial z} = 4x$$

✏️What is the value of the partial derivative of B with respect to z at the point $(\overset{x}{1}, \overset{y}{2}, \overset{z}{3})$?

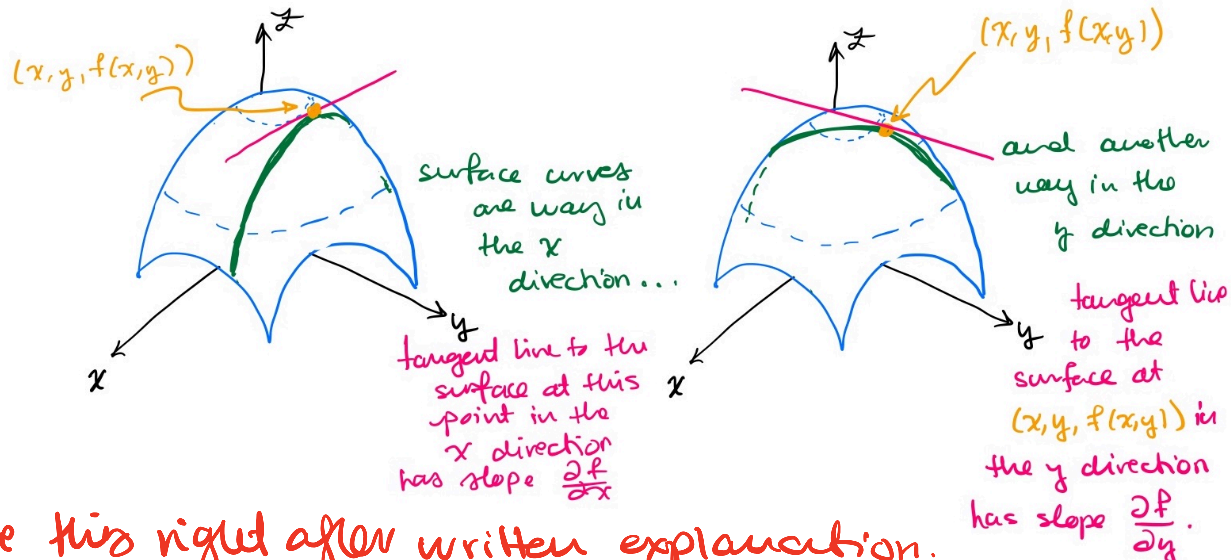
Just plug in $x=1$, $y=2$, $z=3$ in $\frac{\partial B}{\partial z} = 4x \Rightarrow \frac{\partial B}{\partial z} = 4 \cdot 1 = 4$.

*Include explanation about $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$.

OK, but what do partials actually mean?

The slope with respect to one variable if other variables are held constant.

We have a function $f(x,y)$. Its graph is a surface in 3D:



* Move this right after written explanation.

Partial derivatives practice

a. The temperature (in Celsius) across a surface (where x and y are in meters) is described by:

$$T(x,y) = x^2y - 2x + y - 1$$

At what "rate" is temperature changing (with respect to distance):

- In the x direction, at the point (1,3) on the surface?
- In the y direction, at the point (0,2) on the surface?

b. A dragon's breath temperature (T, in degrees Celsius) is modeled as a function of its wingspan (W, in meters) and length (L, also in meters):

$$T(W,L) = 0.41WL + 2.6W^2$$

* Add questions here about writing answer as a complete sentence involving rate of change.

- At what rate is breath temperature changing with respect to length for a dragon that is 4.1m long, with a wingspan of 4.5m?
- At what rate is breath temperature changing with respect to wingspan, for the same dragon?

* Fix slide overflow.

Partial derivatives practice

✏️ Let's see a solution.

a. The temperature (in Celsius) across a surface (where x and y are in meters) is described by:

$$T(x,y) = x^2y - \cancel{2x} + y - \cancel{1}$$

At what "rate" is temperature changing (with respect to distance):

- In the x direction, at the point (1,3) on the surface?
- In the y direction, at the point (0,2) on the surface?

$\frac{\partial T}{\partial x}$ $2yx - 2$

$\frac{\partial T}{\partial x}((1,3)) = 2(3)(1) - 2 = \boxed{4\ °C/m}$

$\frac{\partial T}{\partial y}$ $x^2 + 1$

$\frac{\partial T}{\partial y}((0,2)) = 0^2 + 1 = \boxed{1\ °C/m}$

$\left.\frac{\partial T}{\partial y}\right|_{(0,2)}$

Partial derivatives practice

✏️ Let's see a solution.

a. The temperature (in Celsius) across a surface (where x and y are in meters) is described by:

$$T(x, y) = x^2 y - 2x + y - 1$$

At what "rate" is temperature changing (with respect to distance):

- In the x direction, at the point (1,3) on the surface?
- In the y direction, at the point (0,2) on the surface?

$$\frac{\partial T}{\partial x} = 2xy - 2, \qquad \frac{\partial T}{\partial y} = x^2 + 1$$

$$\left.\frac{\partial T}{\partial x}\right|_{(1,3)} = 2(1)(3) - 2 = 4\ {}^{\circ}\mathrm{C/m}$$

$$\left.\frac{\partial T}{\partial y}\right|_{(0,2)} = (0)^2 + 1 = 1\ {}^{\circ}\mathrm{C/m}$$

Partial derivatives practice

✏️ Let's see a solution.

b. A dragon's breath temperature (T, in degrees Celsius) is modeled as a function of its wingspan (W, in meters) and length (L, also in meters):

$$T(W,L) = 0.41WL + 2.6W^2$$

- At what rate is breath temperature changing with respect to length for a dragon that is 4.1m long, with a wingspan of 4.5m?
- At what rate is breath temperature changing with respect to wingspan, for the same dragon?

(W, L)

$(4.5, 4.1)$

$\frac{\partial}{\partial L} = 0.41W$

$0.41(4.5) = 1.845 \ {}^\circ C/m$

$\frac{\partial}{\partial W} = 0.41L + 2(2.6W)$

$0.41(4.1) + 2(2.6(4.5))$

$= 25.081 \ C^\circ/m$

Partial derivatives practice

✏️ Let's see a solution.

b. A dragon's breath temperature (T, in degrees Celsius) is modeled as a function of its wingspan (W, in meters) and length (L, also in meters):

$$T(W, L) = 0.41WL + 2.6W^2$$

- At what rate is breath temperature changing with respect to length for a dragon that is 4.1m long, with a wingspan of 4.5m?
- At what rate is breath temperature changing with respect to wingspan, for the same dragon?

$$\frac{\partial T}{\partial L} = 0.41W, \qquad \frac{\partial T}{\partial W} = 0.41L + 5.2W$$

$$\left.\frac{\partial T}{\partial L}\right|_{(W=4.5,\, L=4.1)} = 0.41(4.5) = 1.845\ ^{\circ}\mathrm{C/m}$$

$$\left.\frac{\partial T}{\partial W}\right|_{(W=4.5,\, L=4.1)} = 0.41(4.1) + 5.2(4.5)$$

$$= 1.681 + 23.4 = 25.081\ ^{\circ}\mathrm{C/m}$$

Example: higher order & partial derivatives in environmental data science

The **Advection-Dispersion-Reaction Equation** for solute transport models the change in a solute concentration C over time t and x, y, z space, where groundwater is flowing in direction x:

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2} - \upsilon \frac{\partial C}{\partial x} - \lambda RC$$

D_x, D_y, D_z are hydrodynamic diffusion coefficients in the x, y, and z direction.

Let's break it down.

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2} - \upsilon \frac{\partial C}{\partial x} - \lambda RC$$

- **Left-hand-side:** Rate of concentration change (over time)
- **Right-hand-side first 3 terms:** Concentration change due to dispersion in x, y, and z directions
- **Right-hand-side fourth term:** Concentration change due to groundwater transport (in groundwater flow direction, x)
- **Right-hand-side final term:** Reaction term (e.g. biodegradation / abiotic degradation)

Lunch break!

Back at 12:15 ☺

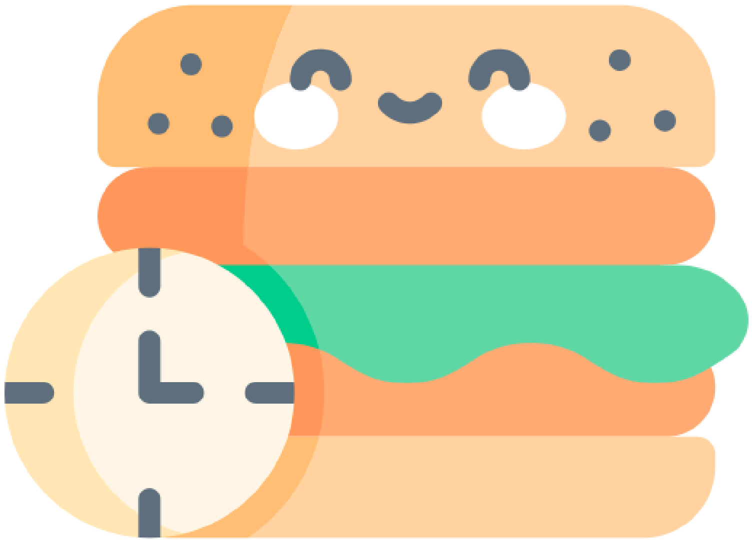


image: Flaticon.com