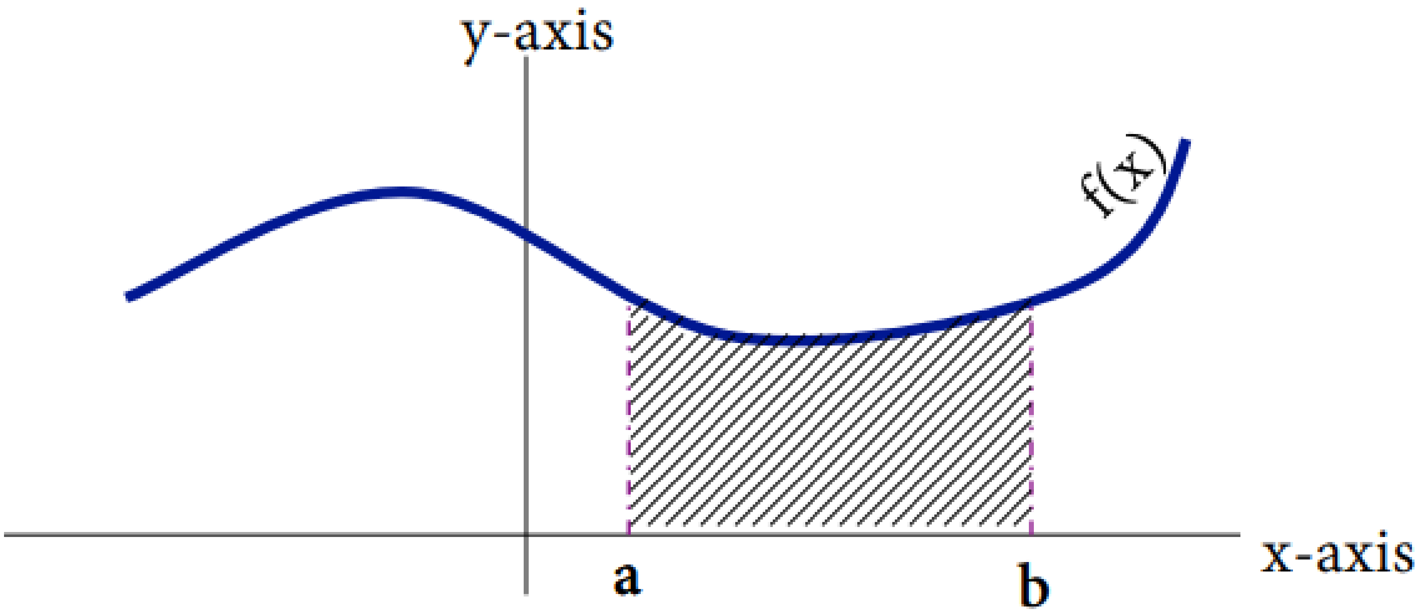
EDS 212: Day 2, Lecture 2

Integration and differential equations

August 4th, 2025

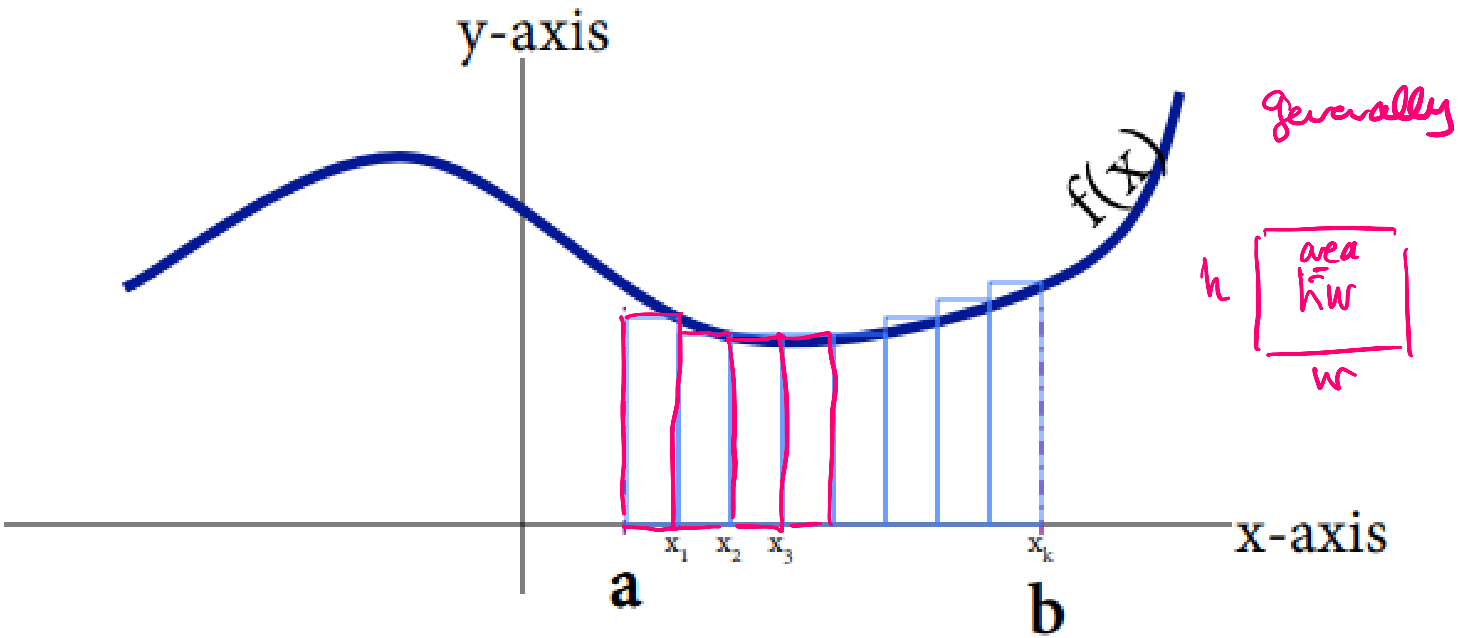
Integration Conceptually

How do we find the area under this curve?



How do we find the area under this curve?

Areas of rectangles are easy to find (base times height!). We can approximate the area by adding the area of rectangles under the curve.

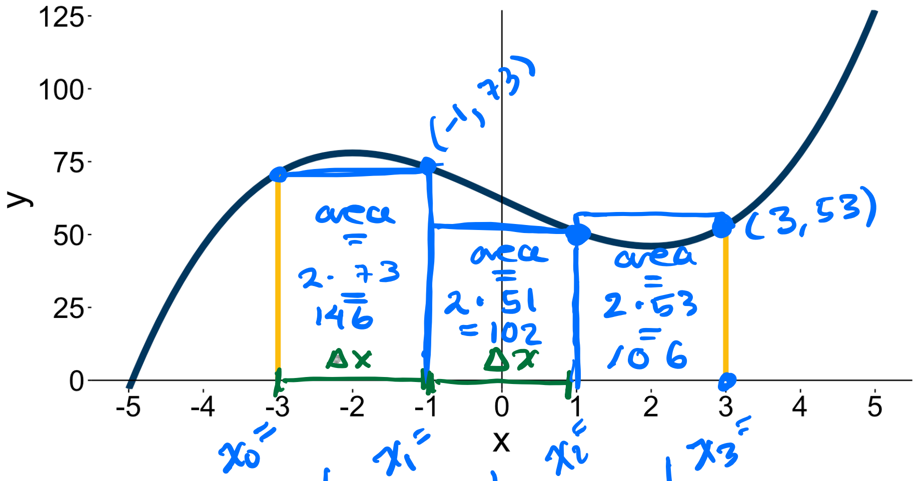


Example

Approximate the area under the curve from $[-3, 3]$ with $\Delta x = 2$ and $k = 3$.

✏️

x	y
$x_0 = -3$	71
$x_1 = -1$	73 $= f(x_1)$
$x_2 = 1$	51 $= f(x_2)$
$x_3 = 3$	53



$R_1 = 146$ $R_2 = 102$ $R_3 = 106$

area approximation $= R_1 + R_2 + R_3$

Riemann sums

1. Divide the interval $[a, b]$ into k subintervals of width Δx.
2. The endpoints of those intervals will give you a series of x values: $x_0, x_1, \ldots, x_k$.
3. Multiply $f(x_1)\Delta x$ to get the the area of the first rectangle.
4. Repeat the same steps above, now starting with x_2 as the bottom right corner of the next rectangle. Repeat until you cover the area with rectangles of width Δx.
5. Sum up the area of all k rectangles.

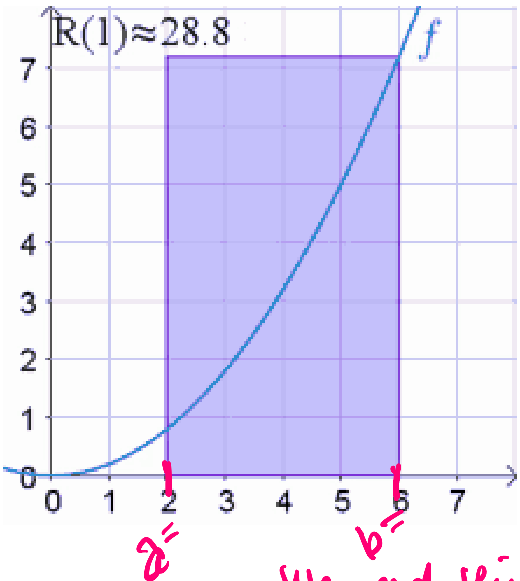
$$R(f(x), \Delta x) = \underbrace{f(x_1)\Delta x + f(x_2)\Delta x + \ldots f(x_k)\Delta x}_{\text{Sum of area of rectangles with width } \Delta(x)}$$

This is called a **Riemann Sum**.

*Figure out rendering here, but probably keep it for information.

Finer and finer Riemann sums

- What if we try to make Δx (the width of the rectangles) really small? So we let $\Delta x \to 0$. This will make the number of rectangles under the curve go to infinity.
- Our estimation of the area will get better and better with more rectangles.



Take the sum of the rectangles as $\Delta x \to 0$

area under the curve is the result of this limit and we denote it by

$$\int_a^b f(x)\,dx.$$

We read this as "the integral from a to b of f(x) with respect to x".

Finer and finer Riemann sums

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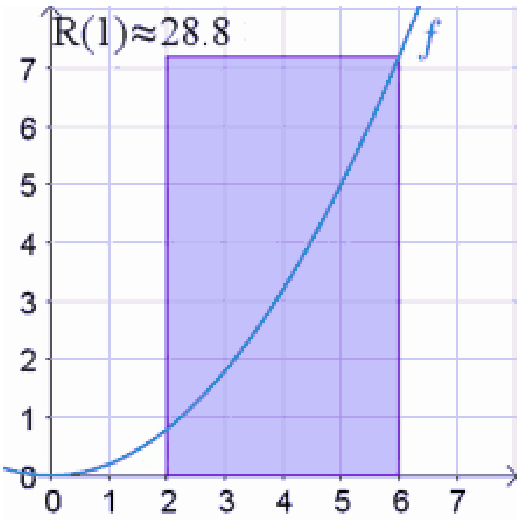


Take the area calculation we did before, but let $\Delta x \to 0$

$$\text{True Area} = \lim_{\Delta x \to 0} R(f(x), \Delta x)$$
$$= \int_a^b f(x)dx$$

We formally call this **the integral of $f(x)$ from a to b with respect to x.**

How do we calculate integrals?

- Taking infinite sums is impossible by hand
- Luckily, integrals and derivatives are related in a very useful way:

if we integrate a derivative, we should recover the original function and vice-versa:

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

: differentiation is the inverse of integration

$$\int f'(x) = f(x) + \underbrace{C}_{\text{integration constant}}$$

: integration is the inverse of differentiation up to a constant

$$\int \frac{df}{dx}\,dx = f(x).$$

To find the integral $\int f(x)\,dx$ we need to find a function $F(x)$ whose derivative is $f(x)$.

How do we calculate integrals?

- Taking infinite sums is impossible by hand
- Luckily, integrals and derivatives are related in a very useful way:

If we integrate a derivative, we should get the same function back as the original vice-versa:

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$ Differentiation is the inverse of integration

$$\int f'(x) = f(x) + C$$ Integration is the inverse of differentiation

* Merge with next slide.

How do we calculate integrals?

- Taking infinite sums is impossible by hand
- Luckily, integrals and derivatives are related in a very useful way:

If we integrate a derivative, we should get the same function back as the original vice-versa:

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) \qquad \text{Differentiation is the inverse of integration}$$

$$\int f'(x) = f(x) + C \qquad \text{Integration is the inverse of differentiation}$$

To find the integral of $\int f(t)dt$ we need to find a function $F(x)$ whose derivative is $f(x)$.

Where did that C come from?

Example

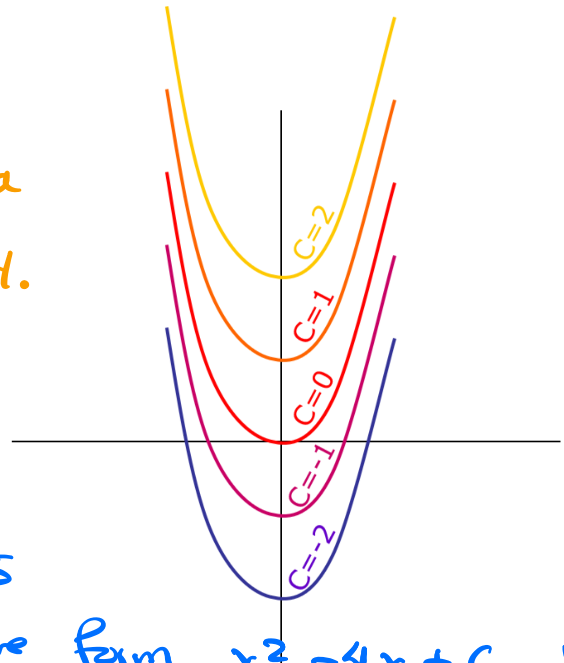
✏️ We want to find $\int 2x-4$

To find $\int 2x-4$ we need to find a function whose derivative is $2x-4$. What could this function be?

Some options: x^2-4x

x^2-4x-2

$x^2-4x+125.5$



generally, any function of the form x^2-4x+C, where C is any number would work.

Solution is: $x^2-4x+C = \int 2x-4$.

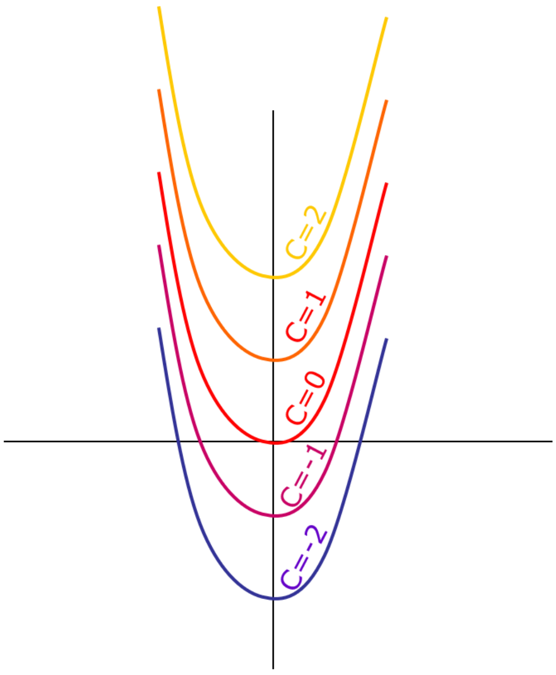
Where did that C come from?

Example

Find $\int 2x - 4$.

To find $\int 2x - 4$ we need to find a function whose derivative is $2x - 4$. What could it be?

- If we integrate, we'll have $\int 2x - 4 = x^2 - 4x + C$.
- There is no way of knowing what the C should be without additional information
- C is the x-intercept and we would have we have to solve for it with an initial value problem.



What is integration useful for?

- Convert from rate of change to total values!

- Solving differential equations
  - i.e $\frac{dy}{dx} = 2x - 4$

- Calculating the area under a curve

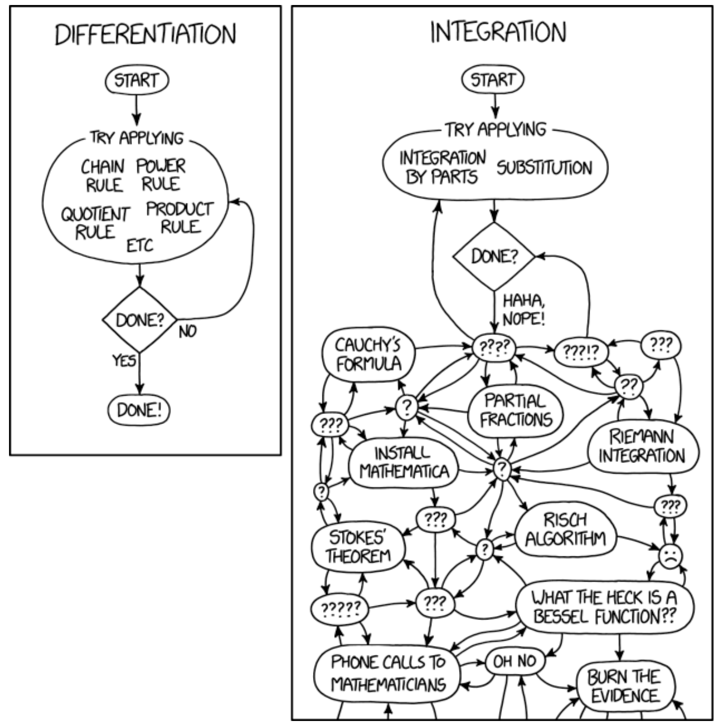
- We'll see some applied exercises too

But also… integration can be tough

Before integrating ask yourself:

1. What am I trying to solve?

2. Does it make sense to take an integral?



Source xkcd comics

Rules of Integration

Notation

When you take an integral, the function in it can be in terms of any variable. This is just notation and it does not affect the result of the integration:

$$\int f(x)dx = \int f(t)dt = \int f(u)du.$$

*Make this into central text.

Integration rules (1)

Sum and Difference Rules

✏️ The integral of a sum is the sum of the integrals:

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

The difference of integrals is the integral of the difference:

$$\int f(x) - g(x)\, dx = \int f(x)\, dx - \int g(x)\, dx$$

Integration rules (1)

Sum and Difference Rules

The integral of a sum is equal to the sum of the integrals:

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

The integral of the difference is equal to the difference of the integrals:

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

Integration rules (2)

Constant Rule

✏️ When a function is multiplied by a constant, the constant can be taken out of the integral:

$$\int c\, f(x)\, dx = c \int f(x)\, dx$$

Integration rules (2)

Constant Rule

When a function is mulitplied by a constant, the constant can be taken out to multiply the integral:

$$\int cf(x)dx = c\int f(x)dx$$

Integration rules (3)

Power Rule

✏️

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

$\int x^n dx$ is a function whose derivative is x^n.

What could it be?

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{n+1}{n+1} x^{n+1-1} = 1 \cdot x^n = x^n.$$

Integration rules (3)

Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \mathrm{n} \neq -1$$

Examples

✏️ Evaluate the following integrals.

a. $\int 4dx$ with respect to x

b. $\int 2x^2 dx$

c. $\int x^3 - 4x dx$

$$\int 4dx = 4x + C$$

$$\int 2x^2 dx = \frac{2x^{2+1}}{2+1} + C = \frac{2x^3}{3} + C$$

$$\int x^3 - 4x^1 dx = \frac{x^4}{4} - \frac{4x^2}{2} + C = \frac{x^4}{4} - 2x^2 + C$$

∂ = partial

δ = delta.

d = a d.

Let's take a 5 minute break

Meet at 2:16!

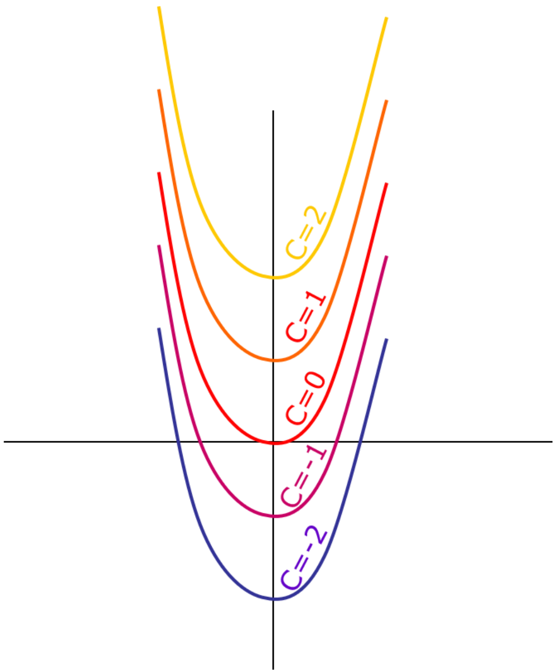
image: Flaticon.com

move this after initial value problems. Add initial value exercise earlier

Initial value problems

Initial value problems

- To find the C term in the integral, we need extra information
- Often times that comes from being given an *initial value*
- This means being given some value of the function, e.g. $y(0) = a$ or $y(12) = b$
- We use this information to solve for what C should be



Example

✏️ The marginal cost of producing x units of a product is:

$$\frac{dC}{dx} = 25 - 0.02x$$

Where C is the cost (in dollars), and x is the number of units produced. Given that producing 2 units of product costs \$10, what is the complete cost equation?

We can integrate to find the original cost function:

$$C(x) = \int \frac{dC}{dx}\, dx = \int 25 - 0.02x \, dx$$

$$= 25x - \frac{0.02x^2}{2} + D.$$

Find value of D by subbing (?) in the given values.

2 units cost \$10 $\Rightarrow C(2) = 10.$

$$\Rightarrow 10 = 25 \cdot 2 - \frac{0.02(2)^2}{2} + D$$

$$\Rightarrow 10 = 50 - 0.04 + D$$

$$\Rightarrow 10 - 50 + 0.04 = -39.96 = D$$

$\therefore C(x) = 25x - \frac{0.02x^2}{2} - 39.96.$

Example

✏️ The marginal cost of producing x units of a product is:

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Where C is the cost (in dollars), and x is the number of units produced. Given that producing 2 units of product costs $10, what is the complete cost equation?

We can integrate to find the original cost function C:

$$C(x) = \int \frac{dC}{dx} dx = \int 25 - 0.02x.$$

Then:

$$\int 25 - 0.02x = 25x - \frac{.02x^2}{2} + D \qquad \text{Solve with Power Rule}$$

$$10 = 25(2) - \frac{0.02(2)^2}{2} + D \qquad \text{Sub in } x = 2$$

$$-39.96 = D$$

$$C(x) = 25x - \frac{0.2x^2}{2} - 39.96$$

Definite integrals

Indefinite vs. definite integrals

✏️ We have been working with <u>indefinite integrals</u> of the form

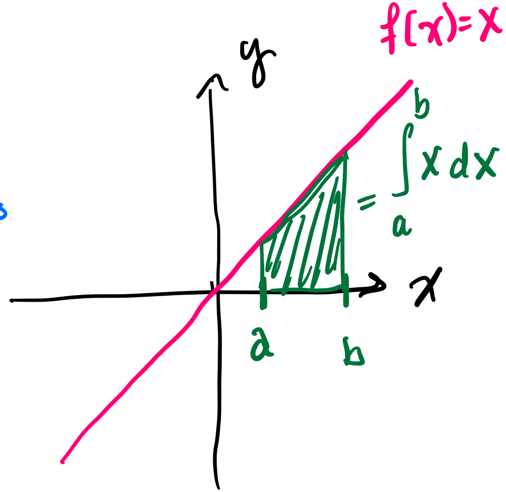
$$\int f(x)\,dx$$

This means there's no integration bounds and initial values are need to find the integration constant.

- <u>Definite integrals</u> have start and end values a and b

$$\int_a^b f(x)\,dx.$$

To find the area under the curve along an interval $[a,b]$, we evaluate the <u>antiderivative</u> at the endpoints and subtract them.



<u>Example</u>: $\int_a^b x^1\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}.$

Indefinite vs. definite integrals

- We have been working with **indefinite integrals** of the form

$$\int f(x)dx$$

This means there are no integration bounds and we need initial values to find integration constants.

* All of these should be combined into a single slide.

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✏️ **Example**

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To find the area under the curve along an interval $[a, b]$, evaluate the antiderivative at the endpoints and subtract them.

✏️ **Example**

$$\begin{aligned} \int_a^b x dx &= \frac{1}{2}x^2\Big|_a^b \\ &= \frac{1}{2}(b)^2 - \frac{1}{2}(a)^2 \end{aligned}$$

Example

✏️ Evaluate the following integral.

$$\int_{-1}^{2} 3x^2\,dx$$

$$\int_{-1}^{2} 3x^2 dx = 3\frac{x^3}{3}\Big|_{-1}^{2} = x^3\Big|_{-1}^{2} = 2^3 - (-1)^3$$

$$= 8 - (-1)$$

$$= \underline{\underline{9}}.$$

What happened with c?

$$= x^3 + c\Big|_{-1}^{2} = (2^3 + c) - ((-1)^3 + c)$$

$$= 8 + c - ((-1) + c) = 8 + 1 + c - c = 9$$

Example

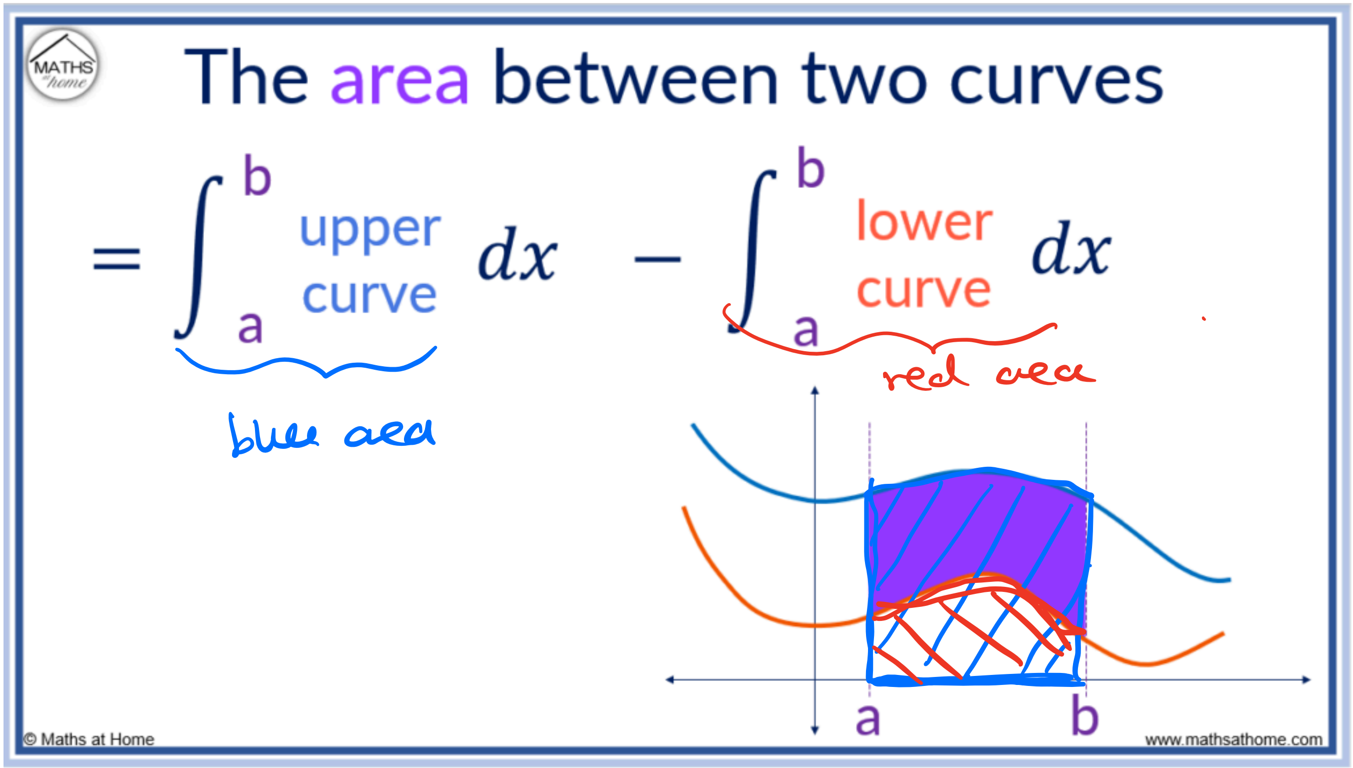
✏️ Evaluate the following integral.

$$\int_{-1}^{2} 3x^2 dx$$

We have that:

$$\begin{aligned}\int_{-1}^{2} 3x^2 dx &= x^3 \Big|_{-1}^{2} \\ &= (2)^3 - (-1)^3 \\ &= 9\end{aligned}$$

Area between two curves



Example

✏️ Find the area of the shaded region in the graph below.

Example

We need to compute

$$\int_0^1 \sqrt{x} - \int_0^1 x^2.$$

Compute each piece carfully:

$$\begin{aligned}\int_0^1 \sqrt{x} &= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 \\ &= \frac{2}{3}(1)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}} \\ &= \frac{2}{3}.\end{aligned}$$

$$\begin{aligned}\int_0^1 x^2 &= \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 \\ &= \frac{1}{3}.\end{aligned}$$

Then subtract the upper from the lower:

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Exercises

Exercises 1

1. Find the integrals of y and g and solve the integral in C.

*A) $y = \frac{3}{x^2}, y(3) = 2$ B) $g(t) = 3t^5 - 2t^3 + 16t - 7$ C) $\int_2^4 \frac{1}{2}x$

$\int g(t) = \frac{t^6}{2} - \frac{t^4}{2} + 8t^2 - 7t + C.$

$\int 3x^{-2}dx = \frac{3x^{-1}}{-1} + C$

$= -\frac{3}{x} + C$

With the initial condition: $y(3) = 2$

$2 = -\frac{3}{3} + C \Rightarrow 2 = -1 + C \Rightarrow C = 3$

$\therefore y(x) = \frac{-3}{x} + 3.$

$\int_2^4 \frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_2^4$

$= \frac{4^2}{4} - \frac{(2)^2}{4}$

$= \frac{16}{4} - \frac{4}{4} = 4 - 1 = 3$

*That (A) exercise is incorrectly stated.

Solution (A)

We have that

$$\begin{aligned}\int \frac{3}{x^2}\,dx &= 3\int x^{-2}\,dx\\ &= 3\left(\frac{x^{-1}}{-1}\right) + C\\ &= -\frac{3}{x} + C\end{aligned}$$

With the initial value $y(3) = 2$ we obtain

$$\begin{aligned}2 &= y(3)\\ &= -\frac{3}{3} + C\\ &= -1 + C\\ 3 &= C.\end{aligned}$$

Therefore the solution is $-\frac{3}{x} + 3$.

Solution (B)

We have that

$$\int \left(3t^5 - 2t^3 + 16t - 7\right) dt = \int 3t^5\, dt - \int 2t^3\, dt + \int 16t\, dt - \int 7\, dt$$

$$= 3\frac{t^6}{6} - 2\frac{t^4}{4} + 16\frac{t^2}{2} - 7t + C$$

$$= \frac{t^6}{2} - \frac{t^4}{2} + 8t^2 - 7t + C$$

Solution (C)

We get that

$$\int_2^4 \frac{1}{2}x\,dx = \frac{1}{2}\int_2^4 x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_2^4$$

$$= \frac{1}{4}\left(4^2 - 2^2\right)$$

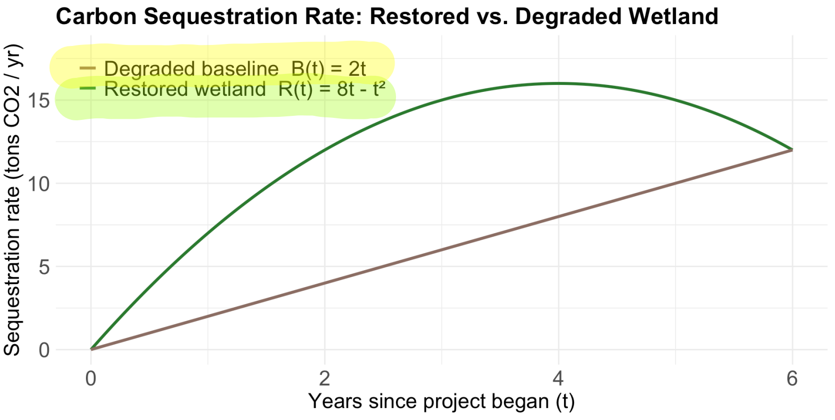
$$= \frac{1}{4}(16 - 4)$$

$$= 3$$

Exercises 2

2. A model for the rate of change in ozone concentrations over time between 1962-1984 is given by $\frac{dC}{dt} = 2t + 20$, where C is the ozone concentration (ppm) and t is the elapsed time in years since 1962. Given that in 1964 the ozone concentration was 30 ppm, what was the ozone concentration in 1982?

3. A coastal wetland restoration project is being evaluated against a "do-nothing" baseline where the wetland is left degraded. Ecologists model the carbon sequestration rate (tons CO_2 / year) as a function of years since the project began, t:



- Restored wetland: $R(t) = 8t - t^2$
- Degraded baseline: $B(t) = 2t$

Both models are valid for $0 \le t \le 6$. What is the **total additional carbon sequestered** by the restored wetland over the 6-year period?

Solution (2)

2. A model for the rate of change in ozone concentrations over time between 1962-1984 is given by $\frac{dC}{dt} = 2t + 20$ where C is the ozone concentration (ppm) and t is the elapsed time in years since 1962. Given that in 1964 the ozone concentration was 30 ppm, what was the ozone concentration in 1982?

✏️

$C(2) = 30$

$\frac{dC}{dt} = 2t + 20$

$\int 2t + 20 \, dt$

$\frac{2t^2}{2} + \frac{20t}{1} + C$

$C(t) = t^2 + 20t + C$

$30 = (2)^2 + 20(2) + C$

$30 = 4 + 40 + C$

$30 = 44 + C$

$-44 \quad -44$

$-14 = C$

$C(t) = t^2 + 20t - 14$

$C(20) = (20)^2 + 20(20) - 14$

786 ppm

Solution (2)

Given:

$$\frac{dC}{dt} = 2t + 20$$

Integrate:

$$\int dC = \int (2t + 20)\, dt$$

$$C(t) = t^2 + 20t + D$$

Initial condition:

$$C(2) = 30 \text{ (since 1964 is 2 years after 1962)}$$

$$30 = 2^2 + 20 \cdot 2 + D$$

$$D = 30 - 44 = -14$$

Thus:

$$C(t) = t^2 + 20t - 14$$

Evaluate at $t = 20$ (1982):

$$C(20) = 20^2 + 20 \cdot 20 - 14$$

$$= 400 + 400 - 14$$

Solution (3)

3. A coastal wetland restoration project is being evaluated against a "do-nothing" baseline where the wetland is left degraded. Ecologists model the carbon sequestration rate (tons CO_2 / year) as a function of years since the project began, t:

- Restored wetland: $R(t) = 8t - t^2$
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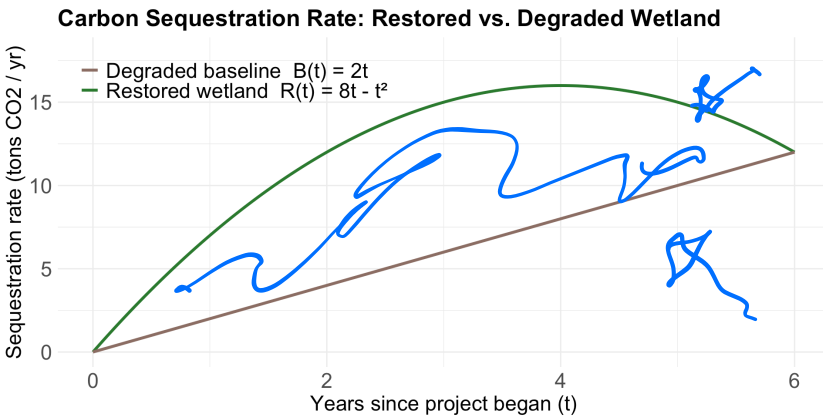


Both models are valid for $0 \le t \le 6$. What is the **total additional carbon sequestered** by the restored wetland over the 6-year period?

✏️

$72 - 36 = 36$ tons $144 - 72 = 72 = 72$

$\int (8t - t^2 - 2t)$ $4(36) - 72$ 72

*Need to account for space for solutions.

$$= \frac{8t^2}{2} - \frac{t^3}{3}\Big|_0^6 4t^2 - \frac{t^3}{3}$$

$$\int_0^6 8t - t^2\,dt = 72$$

$$\int_0^6 2t\,dt = \frac{2t^2}{2}\Big|_0^6 = t^2\Big|_0^6$$

$$= 36$$

$$\int_0^6 8t - t^2\,dt - \int_0^6 2t\,dt$$

$$\int_0^6 8t - t^2 - 2t\,dt$$

$$\int_0^6 6t - t^2$$

Solution (3)

The toal carbon sequestered in each scenario is given by the area under the curve of the corresponding sequestration rate.

So the total *additional* carbon sequesterd is represented by the area between the two sequestration rate curves.

Integrate:

$$\int_0^6 \left[R(t) - B(t)\right] dt = \int_0^6 (6t - t^2)\, dt$$

$$= \left[3t^2 - \frac{t^3}{3}\right]_0^6 = (3(36) - \tfrac{216}{3}) - 0 = 108 - 72 = 36$$

Over the 6-year period, restoring the wetland sequesters an additional **36 tons of CO_2** compared to leaving it degraded.

Differential equations

What is a differential equation?

A **differential equation** is any equation which contains derivatives.

The goal of the differential equation is to find a function that satisfies the equation.

Example

The equation

$$y' = y + x$$

is a differential equation in which we want to find a function $y(x)$ such that its derivative equals the function plus x.

✏️ 1. How would you check that $y(x) = -x - 1$ is a solution for this differential equation?

Differential equations: terms

- **Ordinary differential equation (ODE):** Does not contain partial derivatives

$$\frac{df}{dt} = 3.2 - f(t)$$

Sometimes functions have more than one variable and we can take ***partial derivatives*** *with respect to each variable.*

- **Partial differential equations (PDE):** Differential equations with partial derivatives

Order of an ODE

Order: The order of a differential equation is the highest order for any differential expression in the equation

Example:

$\frac{df}{dt} = 3.2 - f(t)$ is a **first order ordinary differential equation**

Example:

$\frac{\partial^3 x}{\partial t^3} = 2x - 4.5\frac{\partial x}{\partial t}$ is a **third order partial differential equation**

Practice:

Use the terms from the previous slide to describe the following different equations:

$$2.9t^2 - \alpha B = \frac{dB}{dt}$$

$$\frac{\partial^2 f}{\partial x^2} = 1.4 \times 10^{-3} f(x) + 5.2$$

$$\frac{dC}{dt} = 4.1C - 8.0$$

Differential equations help us understand changing environments

- For some natural phenomena it can be easier to describe them in terms of derivatives than give an explicit function.
- By specifying how something changes, we are describing a process or behavior over time.
- Examples:

Describing population dynamics

Groundwater transport of absorbed contaminants

Example: Lotka-Volterra (predator-prey) equations

Prey: $\frac{dV}{dt} = rV - \alpha V P$

Predator: $\frac{dP}{dt} = \beta V P - qP$

Where:

- V is number of prey (e.g. rabbits)
- P is number of predators (e.g. wolves)
- r, α, β, q are positive parameters

What do the different pieces of the equation *mean*?

Interpretation of prey equation:

$$\frac{dV}{dt} = rV - \alpha VP$$

The pieces:

- $\frac{dV}{dt}$: Rate of growth / decline in prey abundance
- rV: Population growth (without loss due to predation (assumes predators are the only force limited the growth of the prey population), where r is the intrinsic rate of increase)
- $-\alpha VP$: Population loss due to predation (where α is the capture efficiency of the predator)

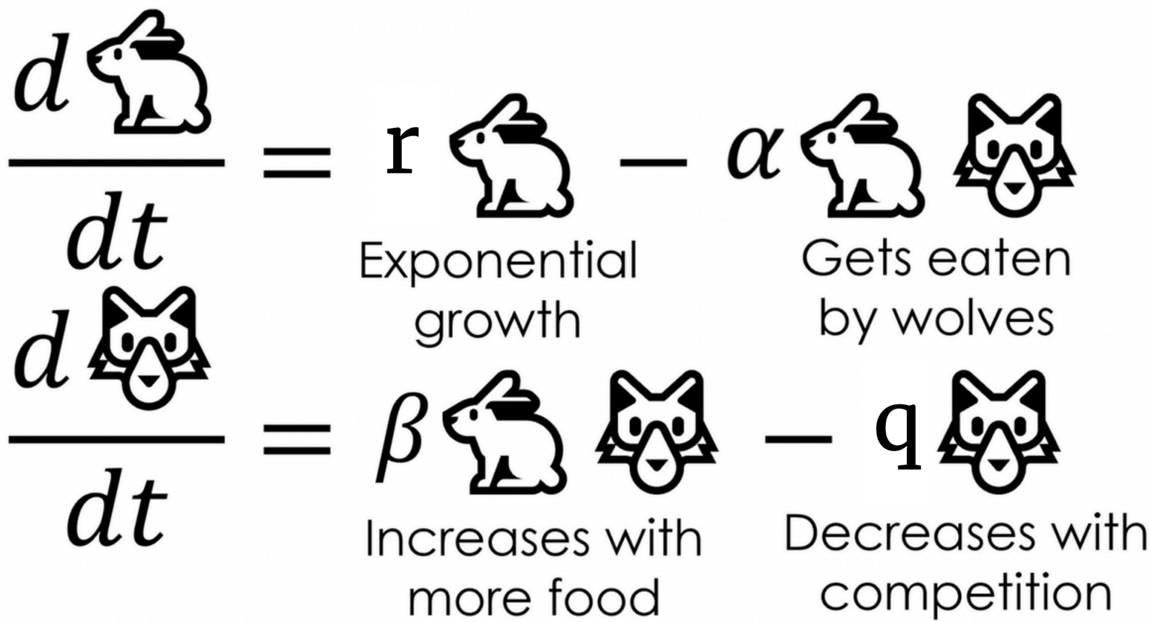
Interpretation of predator equation:

$$\frac{dP}{dt} = \beta V P - qP$$

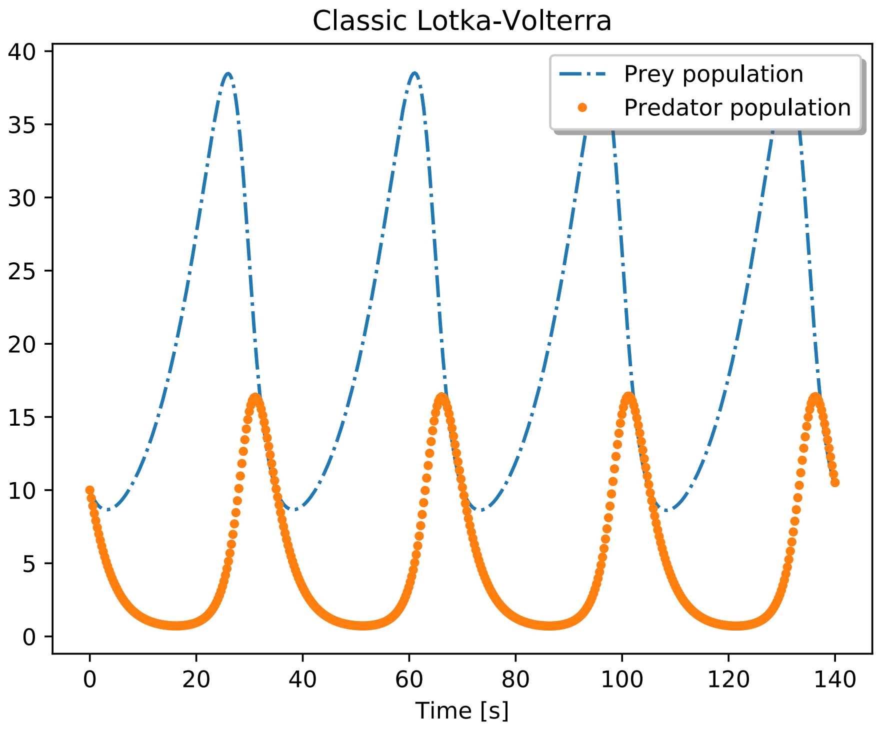
The pieces:

- $\frac{dP}{dt}$: Rate of growth / decline in predator population
- $\beta V P$: Predator population growth (where β is a measure of conversion efficiency, i.e. the ability of predators to convert each new prey into additional per capita growth rate for the predator population)
- $-qP$: Predator population loss (where q is the per capita death rate; positive population growth only happens when prey are present)

Or, in pictures:



Finding $V(t)$ and $P(t)$?



Some differential equations can be solved analytically:

Some differential equations can be solved analytically:

$$\frac{dy}{dx} = y$$

Separate variables and integrate both sides:

$$\int \frac{1}{y} dy = \int 1 dx$$

Yielding:

$$ln(y) = x \text{ or } y = e^x$$

Solving differential equations numerically

Find *approximate* solutions to differential equations when finding an analytical solution would be really challenging (...which is pretty often).

Instead, computers can numerically approximate solutions by predicting nearby values based on the *slope*.

There are many methods for solving differential equations numerically. You'll do it programatically later on.

Wrapping up...

What we covered today

- 🔎 Limits
- 🖋️ Derivatives and rate of change
- 🛤️ Tangent lines
- 📝 Rules for differentiation
- 🔷 Constant rule
- 💪 Power rule
- ➕ Sum and difference rule
- ~~✖️ Product rule~~
- 📈 Higher order derivatives
- 🌓 Partial derivatives
- ∫ Integrals
- 🧮 Riemann sums
- 📐 Rules of integration
- ✏️ Initial value problems
- 🧾 Definite and indefinite integrals
- 📏 Area between two curves
- 🌊 A bit of ODEs

* Update list