# EDS 212: Day 3, Lecture 2

*Vectors and matrices*

August 5th, 2025

# Summation Notation

# Summation notation

✏️ is a compact way to write the sum of a sequence of values.

Idea: instead of writing long sums like:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$

we can write

upper bound

$$\sum_{i=1}^{10} x_i \, .$$

the expression we iterate over

lower bound

Read this as the sum from i=1 to 10 of x sub i.

Few components:

- $\sum$ = capital sigma, a Greek letter! means sum everything that follows
- $i$ = index, counter that steps through values

# Summation notation

Summation notation is a compact way to write the sum of a sequence of values.

The idea is that, instead of writing long sums like

$$x_1 + x_2 + \cdots + x_n$$

we can write the same in a more compact form:

$$\sum_{i=1}^{n} x_i \,.$$

This is read as *the sum of $x_i$ from i equals one to $n$*.

There's a few components to this notation:

# Examples

**Example 1:** Evaluate $\sum_{i=1}^{5} i^2$.

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

**Example 2:** Evaluate $\sum_{n=1}^{3} (2n + 1)$.

$$\rightsquigarrow \sum_{n=1}^{3} 2n + 1$$

inside the sum?

$$\sum_{n=1}^{3} (2n+1) = (2\cdot 1 + 1) + (2\cdot 2 + 1) + (2\cdot 3 + 1) = \underline{\underline{15}}$$

# Examples

**Example 1:** Evaluate $\sum_{i=1}^{5} i^2$.

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

**Example 2:** Evaluate $\sum_{n=1}^{3} (2n + 1)$.

$$\sum_{n=1}^{3} (2n + 1) = \big(2(1) + 1\big) + \big(2(2) + 1\big) + \big(2(3) + 1\big) = 3 + 5 + 7 = 15$$

# Exercises

✏️ Try these out individually first.

**1)** Expand and evaluate: $\sum_{i=3}^{6} 3i$

**2)** Expand and evaluate: $\sum_{i=2}^{5} (i^2 - 1)$

**3)** Complete the following using sigma notation:

$$x_1 + x_2^2 + x_3^3 + \cdots + x_n^n = \sum$$

**4)** A climate model predicts daily high temperatures $\hat{y} = [22, 25, 20]$ °C. Observed values are $y = [24, 23, 21]$ °C. Calculate the mean squared error (MSE) using the formula:

$$\text{MSE} = \frac{1}{n} \sum_{i=1}^{n} (\hat{y_i} - y_i)^2 .$$

# Solutions

✏️

**1)** Expand and evaluate: $\sum_{i=3}^{6} 3i$ $= 3\cdot 3 + 3\cdot 4 + 3\cdot 5 + 3\cdot 6 = \underline{\underline{54}}.$

**2)** Expand and evaluate: $\sum_{i=2}^{5} (i^2 - 1)$ $= (2^2-1) + (3^2-1) + (4^2-1) + (5^2-1) = 50$

**3)** Complete the following using sigma notation:

$$x_1 + x_2^2 + x_3^3 + \cdots + x_n^n = \sum_{i=1}^{n} x_i^i = \sum_{m=1}^{1} x_m^m$$

$$? \sum_{i=1}^{n} x^i = x^1 + x^2 + \ldots + x^n.$$

# Solutions

✏️

**4)** A climate model predicts daily high temperatures $\hat{y} = [22, 25, 20]$ °C. Observed values are $y = [24, 23, 21]$ °C. Calculate the mean squared error (MSE) using the formula:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(\hat{y_i} - y_i)^2.$$

start at 0, $n=3$

$\rightsquigarrow (y_0, y_1, y_2)$

$\frac{1}{n}\sum_{i=0}^{n-1}(\hat{y}_i - y_i)^2.$

$\hat{y} = (22, 25, 20) = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$

$y = (24, 23, 21) = (y_1, y_2, y_3)$

$n = 3$

Compute squared errors first:

$\underbrace{(y_1-\hat{y}_1)^2 + (y_2-\hat{y}_2)^2 + (y_3-\hat{y}_3)^2}_{\sum_{i=1}^{3}(y_i-\hat{y}_i)^2 = 9} = 4 + 4 + 1 = 9$

$\Rightarrow \frac{1}{3}\sum_{i=1}^{3}(y_i - \hat{y}_i) = \frac{1}{3}\cdot 9 = 3°C$

# Solutions

**1)**

$$\sum_{i=3}^{6} 3i = (3 \cdot 3) + (3 \cdot 4) + (3 \cdot 5) + (3 \cdot 6)$$
$$= 3(3 + 4 + 5 + 6)$$
$$= 54$$

**2)**

$$\sum_{i=2}^{5} (i^2 - 1) = (4 - 1) + (9 - 1) + (16 - 1) + (25 - 1)$$
$$= 3 + 8 + 15 + 24$$
$$= 50$$

**3)**

$$x_1 + x_2^2 + \cdots + x_n^n = \sum_{i=1}^{n} x_i^i$$

**4)** The formula is $\mathrm{MSE} = \frac{1}{n} \sum_{i=1}^{n} (\hat{y_i} - y_i)^2$. Compute each squared error first:

$$(\hat{y_1} - y_1)^2 = (22 - 24)^2 = 4$$
$$(\hat{y_2} - y_2)^2 = (25 - 23)^2 = 4$$
$$(\hat{y_3} - y_3)^2 = (20 - 21)^2 = 1$$

Then apply the formula:

$$MSE = \frac{4 + 4 + 1}{3} = 3\ °C$$

# Vectors and matrices

# Linear algebra introduction

Linear algebra “branch of mathematics concerning linear equations” (Wikipedia), sometimes also described as the math of vectors & matrices.

It is a fundamental part of data science (and how computers understand & process data), and useful for describing environmental processes.

# The building blocks of linear algebra

✏️
- a scalar : a value without direction, represents magnitude.
  a fancy word for a number : 8, 1, $\lambda$, ...

- a vector : an ordered list of values, represent
  magnitude and direction (physics)
  values for an observation (data science)
  ex: $(1, 3, 5)$ or $(x_1, x_2, x_3)$.

- a matrix : an array of values made up of rows and
  columns.

ex: $\begin{pmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{pmatrix}$ or $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

# The building blocks of linear algebra

- **scalar:** a value without direction, representing magnitude. For our purposes, a number:

$$8 \text{ or } \lambda.$$

- **vector:** an ordered list of values, representing magnitude *and* direction (physics) or values for an observation or variable (data science). For example,

$$(1, 3, 5) \text{ or } (x_1, x_2, x_3).$$

- **matrix:** am array of values made up rows and columns. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

# Applications of linear algebra in environmental sciences

- Dimensional reduction
- Population matrix models
- Optimization
- Array programming / vectorized code
- Machine learning

- Remote sensing (rasters)
- Image analysis.

# Let's start with vectors

**Where are vectors in EDS?**

# Let's start with vectors

## Where are vectors in EDS?

## *Everywhere.*

Vectors are lists of values used to describe different features or variables of interest. For example, if you are trying to model *fish size* based on length (cm) and mass (g), then for a fish with length 32 cm weighing 281 g, you might describe that by:

$$(32, 281)$$

Sometimes vectors are represented with an arrow over the vector name:

$$\vec{x} = \vec{x} = (32, 281)$$

✏️ Often we think about an "abstract" vector or observed variables or vector of features with n coordinates

$$x = (x_1, x_2, x_3, \ldots, x_n).$$

# Let's start with vectors

## Where are vectors in EDS?

### *Everywhere.*

Vectors are lists of values used to describe different features or variables of interest. For example, if you are trying to model *fish size* based on length (cm) and mass (g), then for a fish with length 32 cm weighing 281 g, you might describe that by:

$$(32, 281)$$

Sometimes vectors are represented with an arrow over the vector name:

$$\vec{x} = (32, 281)$$

Often, we will think of an "abstract vector" $x$ with $n$ **coordinates**:

$$x = (x_1, x_2, \ldots, x_n).$$

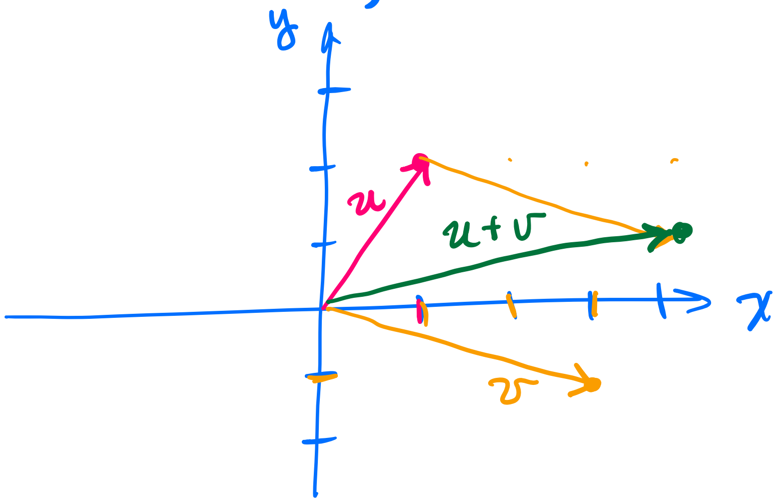
# Vector addition & subtraction

Just add or subtract the corresponding coordinates.

✏️ $u = (1, 2)$ and $v = (3, -1)$ then

$u + v = (1+3\ ,\ 2-1) = (4, 1).$

graphically:



# Vector addition & subtraction

Just add or subtract the corresponding coordinates.

If: $u = (1, 2)$ and $v = (3, -1)$, then:

$$u + v = (1 + 3, 2 - 1) = (4, 1)$$

What does this look like graphically? **Let's draw it!**

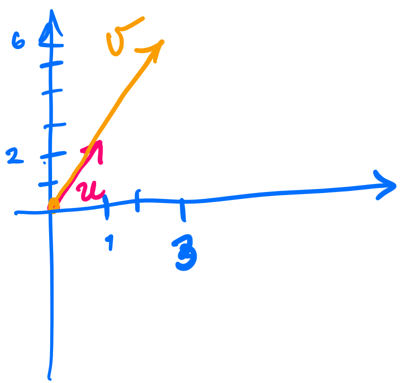
# Scalar multipliers

You can multiply any vector by a scalar (constant). This will not change the *direction* of the vector – it will only change the *magnitude* of the vector.

✏️ example $u = (1, 2)$

scalar : $3$

$$3u = (3\cdot 1,\ 3\cdot 2) = (3, 6)$$



# Scalar multipliers

You can multiply any vector by a scalar (constant). This will not change the *direction* of the vector - it will only change the *magnitude* of the vector.

**Example:** $u = (1, 2)$

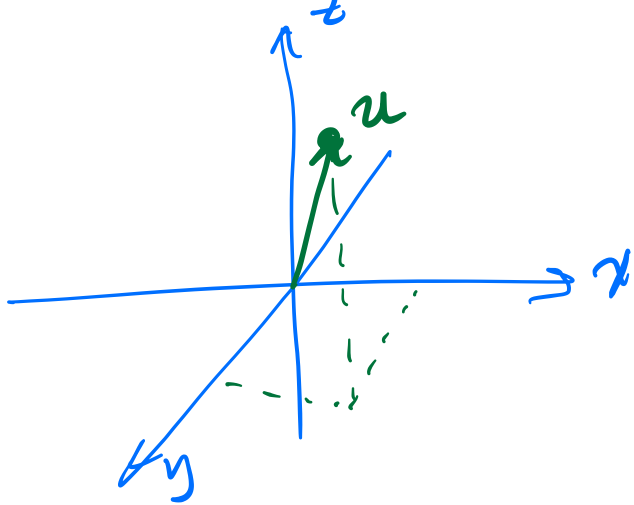
$$w = 3u = (3 * 1, 3 * 2) = (3, 6)$$

What does this look like graphically? **Let's draw it!**

# Vectors with > 3 coordinates

What about a vector with more than two coordinates? More than three?

Is as valid as describing a “point” in multivariate space as a vector with two “coordinates” – it’s just difficult for us to visualize and conceptualize since our brain only happily deals with 3 dimensions.



# Dot product

✏️ For vectors

$$x = (x_1, \ldots, x_n) \quad \text{and} \quad y = (y_1, \ldots, y_n)$$

their <u>dot product</u> is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i .$$

We can think about it as a measure of how much the vectors $x$ and $y$ point in the same direction.

# Dot product

For vectors

$$x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n)$$

their **dot product** is:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

**In words:** The dot product is the sum of coordinates of each vector multiplied together. It is a measure of how close the vectors "point" in the same direction

# Exercise

For vectors $x = (x_1, \ldots, x_n)$ $y = (y_1, \ldots, y_n)$ their **dot product** is:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \quad = |x| \cdot |y| \cdot \cos(\theta)$$

Find the dot product of $a = (2, -1, 0)$ and $b = (9, 3, -4)$:

$(x_1, x_2, x_3)$ $\quad$ $(y_1, y_2, y_3)$

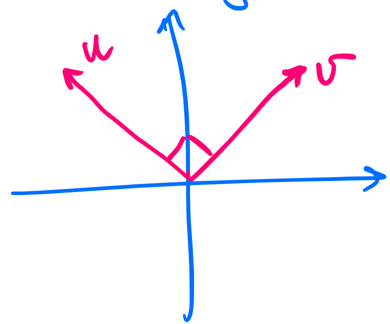
✏️

$$a \cdot b = x_1 \cdot y_1 + x_2 y_2 + x_3 \cdot y_3$$

$$= 2 \cdot 9 + 3(-1) + (-4) \cdot 0$$

$$= \underline{\underline{15}}.$$

orthogonal:



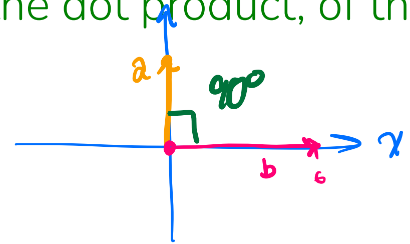
# Exercise

For $a = (2, -1, 0)$ and $b = (9, 3, -4)$:

$$a \cdot b = (2)(9) + (-1)(3) + (0)(-4) = 15$$

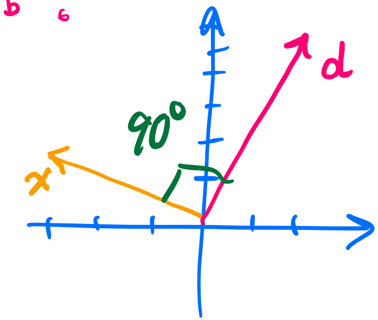
# What happens when we have orthogonal vectors?

✏️ Sketch a quick graph, then find the dot product, of the following vector combinations:




1. $a = (0, 4)$ and $b = (6, 0)$

   $a \cdot b = 0$




2. $x = (-3, 1)$ and $d = (2, 6)$

   $x \cdot d = 0$

What is the value of the dot product for orthogonal vectors?

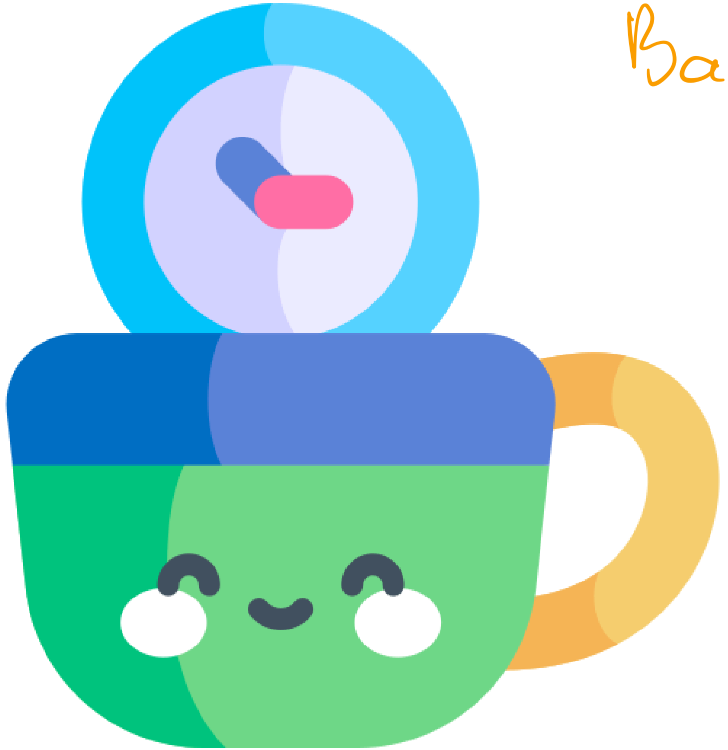
if two vectors are orthogonal, $u \cdot v = 0$.

# More on vector fundamentals:

**Optional:** watch 3Brown1Blue's great 10 min recording on Vectors (Ch 1 Essense of Linear Algebra).

Vector addition & scalar multiplication are the basis of most linear algebra!

# Let's take a 5 minute break



Back at
2:20.

image: Flaticon.com

# Matrices

# Matrices

A matrix is an array of values (has rows and columns). For example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

- **Dimensions:** the size of the matrix, in rows x columns ($m$ x $n$)
- **Elements:** values in a matrix, often denoted symbolically with a subscript where the first number is the *row* and the second number is the *column* (e.g. $a_{23}$ indicates the element in row 2, column 3)

row column

# Matrix algebra (add / subtract)

Add or subtract the corresponding elements (by matrix position) to create a new matrix of the same dimensions.

✏️

ex:

$$\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 5 & 8 \end{pmatrix}.$$

# Matrix algebra (add / subtract)

Add or subtract the corresponding elements (by matrix position) to create a new matrix of the same dimensions.

$$\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 8 \end{bmatrix}$$

# Scalar multiplication

To multiply a matrix by a *scalar*, multiply each element in the matrix by the scalar to get a scaled matrix of the same dimensions.

For example:

✏️

$$6 \cdot \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 6\cdot 1 & 6(-3) \\ 6\cdot 0 & 6\cdot 2 \end{pmatrix} = \begin{pmatrix} 6 & -18 \\ 0 & 12 \end{pmatrix}.$$

# Scalar multiplication

To multiply a matrix by a *scalar*, multiply each element in the matrix by the scalar to get a scaled matrix of the same dimensions.

For example:

$$6 \times \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -18 \\ 0 & 12 \end{bmatrix}$$

# Recall: dot product

The dot product of two vectors is the ~~sum~~ product of their elements ~~multiplied~~ added:

For $u = (1, 5)$ and $v = (2, -3)$:

$$u \cdot v = (1)(2) + (5)(-3) = -13$$

# Matrix multiplication

We find the *dot product* of row · column vectors:

✏️

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

# Matrix multiplication

We find the *dot product* of row · column vectors:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

# Practice problems

✏️ Find the product of these two matrices

$$= \begin{pmatrix} -4 & 13 \\ 24 & -12 \end{pmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 5 \\ 6 & -3 \end{bmatrix} = \underset{A}{?}$$

# Matrices with unequal dimensions

What would be the dimensions of the product if you were multiplying the following?

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \times \begin{bmatrix} G & H \\ I & J \\ K & L \end{bmatrix} =$$

# Let's try one!

$2 \times 3$ $\qquad\qquad$ $3 \times 2$

\# of cols of 1st one = # rows of 2nd on

$$\begin{bmatrix} 0 & 2 & -1 \\ 3 & 4 & -2 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 5 & 1 \\ -3 & 6 \end{bmatrix} = ?$$

$$= \begin{pmatrix} 0(-1)+2(5)+(-1)(-3) & (0)(0)+2\cdot 1+(-1)6 \end{pmatrix}$$

$3(-1)+4(5)+(-2)(-3) \quad (3)0+4\cdot 1+(-2)6$

# Diagonal matrix

$= \begin{pmatrix} 13 & -4 \\ 23 & -8 \end{pmatrix}$

A **diagonal matrix** is (almost always) a square matrix ($m = n$) where only elements on the diagonal are non-zero values.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

# What happens when we multiply a matrix by a diagonal matrix?

A diagonal matrix is also called a *scaling matrix* because it scales rows proportionally, but not by the same value:

✏️

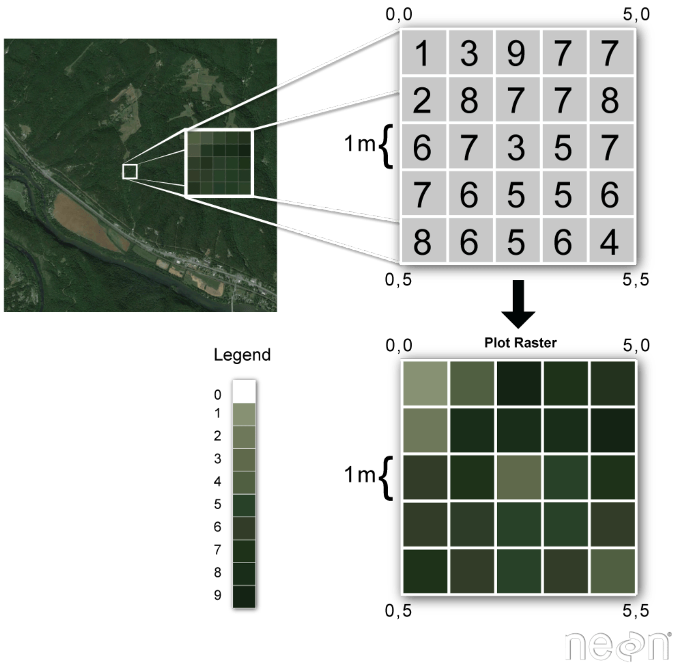
# What happens when we multiply a matrix by a diagonal matrix?

A diagonal matrix is also called a *scaling matrix* because it scales rows proportionally, but not by the same value:

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 8 & 4 \end{bmatrix}$$

# Where do matrices come up in EDS?

Rasters (matrix like)



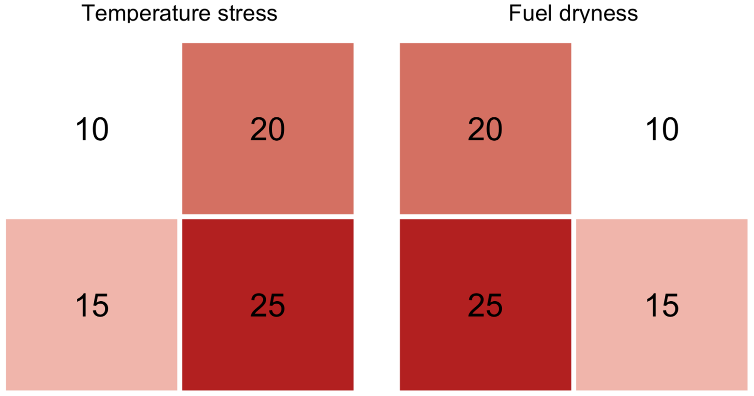

Source: Neon

# Exercise: combining rasters

A raster is just a matrix where each cell holds a value for a location on a grid. Land managers often combine several raster layers into a single index by **scaling** each layer (multiplying by a weight) and **summing** the results.

✏️ Calculate the fire risk index for each cell:

$$\text{Fire risk} = 0.6 \times \text{Temperature stress} + 0.4 \times \text{Fuel dryness}$$



$$\text{Risk} = 0.6\begin{bmatrix}10 & 20\\ 15 & 25\end{bmatrix} + 0.4\begin{bmatrix}20 & 10\\ 25 & 15\end{bmatrix}$$

$$= \begin{bmatrix}6 & 12\\ 9 & 15\end{bmatrix} + \begin{bmatrix}8 & 4\\ 10 & 6\end{bmatrix}$$

$$= \begin{bmatrix}14 & 16\\ 19 & 21\end{bmatrix}.$$

Rasters let you multiply coordinate by coordinate:

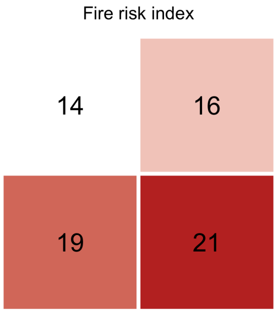
$$\begin{bmatrix}10 & 20\\ 15 & 25\end{bmatrix} \times \begin{bmatrix}20 & 10\\ 25 & 15\end{bmatrix} = \begin{bmatrix}200 & 200\\ 375 & 375\end{bmatrix}$$

# Exercise: solution

💡 Let's see a solution!

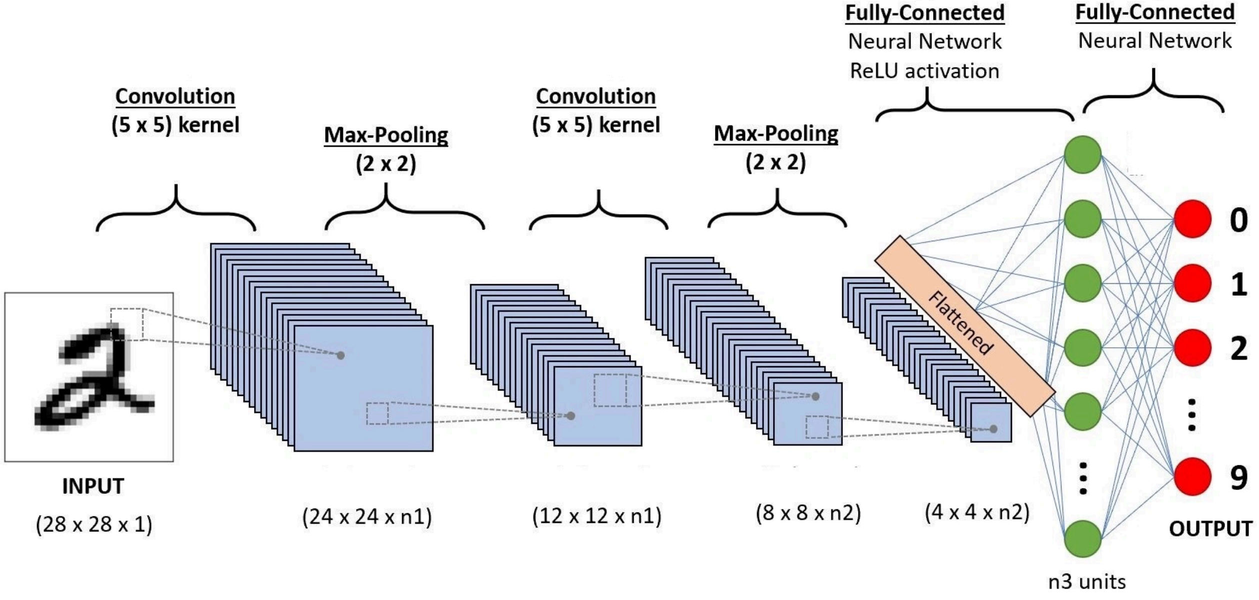
$$\text{Fire risk} = 0.6 \times \text{Temperature stress} + 0.4 \times \text{Fuel dryness}$$

- Top-left:
  $0.6(10) + 0.4(20) = 6 + 8 = 14$
- Top-right:
  $0.6(20) + 0.4(10) = 12 + 4 = 16$
- Bottom-left:
  $0.6(15) + 0.4(25) = 9 + 10 = 19$
- Bottom-right:
  $0.6(25) + 0.4(15) = 15 + 6 = 21$



# Where do matrices come up in EDS?

Machine learning (EDS 232)




Adapted from What is the Convolutional Neural Network Architecture?

# Wrapping up...

# What we covered today

- 📝 **Practice all day 1 and 2 topics**
- 🌊 **A bit of ODEs**
- ➕ **Summation notation**
- 🔷 **Vector sum and product**
- 🌓 **Dot product**
- 🧮 **Matrix algebra**